THE

MATHEMATICAL GAZETTE

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AUTOMATIC CALCULATING MACHINES.*

BY D. R. HARTREE, F.R.S.

1. *Automatic digital calculating machines.*

When we write a number in the ordinary way, such as 673, the symbols such as 6, 7, 3 in this example stand for what we call the *digits* of the number, and a piece of mechanical or electrical equipment which operates with and records the discrete digits of each number is often called a *digital calculating machine*. In the past twelve years there has been a remarkable development of such machines with two important features. First, they can carry out lengthy and intricate calculations quite automatically once they have been supplied with a specification, in suitable form, of the calculation to be carried out, and secondly, they are very versatile, so that the same machine can be used for many quite different kinds of calculation, for example for evaluating values of a function from its power-series expansion, for solving large systems of linear simultaneous algebraic equations, for finding the characteristic values of matrices, for the step-by-step integration of ordinary differential equations. To express these two features, such machines are sometimes called *general-purpose, automatic*, digital calculating machines, and my purpose is to give an account of some aspects of such machines, both of the machines themselves and of the way in which they are used. I can only touch on some aspects, since there is not nearly enough time in a single lecture to deal with the subject fully.†

2. *Babbage's "Analytical Engine".*

Although it is only about eight years since the first such machine was put into operation, the idea is not a new one. It is about 115 years old and due in principle to Charles Babbage, whose projected machine ([2, 3, 15]) which he called the "analytical engine" was to be a general-purpose digital calculating machine, automatic except for one thing, namely, the use of tables of functions, for which the services of an operator were required.

The "analytical engine" was to consist of three main components; one,

* A paper to the Annual Meeting of the Mathematical Association, January, 1950.

† For a fuller general survey see reference 10; for more detailed accounts of individual machines, see references 1, 4, 8, 9, 11, 12, 13, 14 and 20.

which Babbage called the " store ", in which numerical information could be recorded on a set of registers ; another, which he called the " mill ", in which arithmetical operations could be carried out on numbers transferred to it from the store, and another, to which he did not give a name, which selected the registers from which numbers were to be transferred from the store, the operation to be carried out on them, and the register to which the result was to be transferred.

This control of the operation of the machine was to be carried out through a sequence of punched cards. Plungers passing through holes in the cards were to select the registers for the transfers to and from the mill, and the operation to be carried out there.

3. *General aspects of automatic digital calculating machines.*

Consideration of the organisation of a calculation done by a human computer with the assistance of a desk machine will show what facilities have to be provided for carrying out the same calculation automatically, and the conclusions can be illustrated by the organisation of Babbage's " analytical engine ".

A schematic diagram of the organisation of a hand computation is shown in Fig. 1*a*. The computer has available the desk machine for arithmetical operations, a work sheet for keeping note of data of the calculation and of the required sequence of arithmetical operations, and for recording intermediate results, and he controls the transfer of numbers either way between the desk machine, work sheet and tables. He also controls the operation of the desk machine, and takes from the work sheet the specification of the transfers and operations to be carried out.

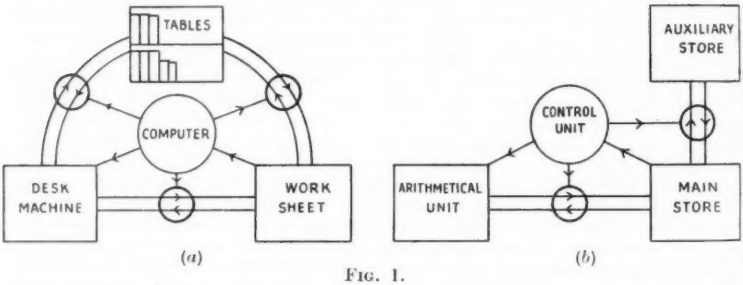

FIG. 1.

(*a*) Organisation of a hand calculation carried out with the aid of a desk machine.
(*b*) Organisation of a calculation with an automatic digital machine.

An automatic machine can be thought of as having a similar organisation (see Fig. 1*b*). It must have an *arithmetical unit* (the " mill " of Babbage's Analytical Engine) in which arithmetical operations can be carried out, a *store* for numerical data, intermediate results, and operating instructions, and a *control unit* to take the place of the human computer who controls the sequence of operations in a hand calculation. The machine also needs input and output equipment for receiving operating instructions and numerical data from the outside world and for presenting its results. The store may be arranged to include the functions of the work sheet and the books of tables in a hand calculation, and it may be convenient to divide it into two parts, a " main store " with direct access to the arithmetical unit, and an " auxiliary

store " to which groups of numbers or instructions can be transferred when in the course of a calculation they will not be wanted for some time.

4. *Representation of numbers in the machine.*

It is often convenient to represent the digits of a number by physical elements each of which has only two distinct indications, which are mutually exclusive. For example, we may use a set of locations on a card, at each of which either a hole or no hole may be punched, or a set of relays each of which may be either " on " or " off ", or a set of signals each consisting of the presence or absence of an electrical pulse. The use of such elements suggests the use of a binary representation of numbers within the machine, though it does not impose it of necessity since it is possible to code decimal numbers onto such elements in several ways. Several machines do work with numbers expressed, within the machine, in scale of two ; nearly all such machines accept numerical information and produce results expressed in scale of ten, the decimal-binary and binary-decimal conversions being effected by the machine as part of the input and output procedure.

The digits of a number can be represented by the simultaneous states of a number of elements, or by the successive states in time of a single element. For example, the digits of a number expressed in scale of two may be represented by *simultaneous* indications, each consisting of the presence or absence of a pulse, on a set of lines, one for each digital position, feeding into a storage or arithmetical unit (Fig. 2*a*), or by a time *sequence* of pulses on a *single* line (Fig. 2*b*). The former is called a " parallel " and the latter a " serial " repre-

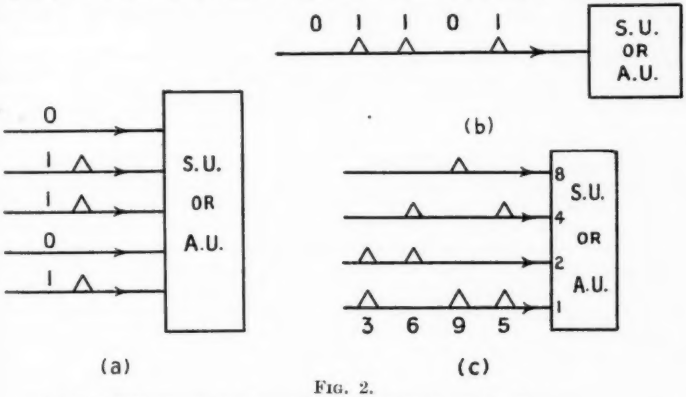

FIG. 2.

(*a*) Parallel treatment of a number expressed in scale of two.
(*b*) Serial treatment of a number expressed in scale of two.
(*c*) Series-parallel treatment of a number expressed in scale of ten ; parallel treatment of binary form of each decimal digit, serial treatment of successive decimal digits.

S.U. or A.U. = storage unit or arithmetical unit.

sentation, and correspondingly we can have " parallel " or " serial " operation according as the digits of a number are operated on simultaneously or successively. It is also possible to use a mixed series-parallel representation. For example, in a decimal machine a parallel binary representation can be used for each decimal digit, the separate decimal digits being treated serially (Fig. 2*c*).

In the process of addition, carry-over is from the less significant to the more significant digital position. For this reason it is most convenient in a serial machine to have the numbers represented with the least significant digit leading.

5. Arithmetical operations.

In a decimal machine there are two ways of carrying out the process of addition. One consists of *counting*, that is, of successive additions of *units*; the other consists of the use of an addition *table*, by which, in adding 3 to 4 for example, we go directly to the result 7 without going through 5 and 6 as intermediate stages. In a binary machine there is no distinction between the two methods; both involve the four relations:

$$0+0=0, \qquad 1+0=1,$$

$$0+1=1, \qquad 1+1=10$$

(the last of these to be read, of course, in scale of two, in which 10 stands for a *two* and no units). In a serial binary machine, the most that is being done at any moment is the addition of a single digit 0 or 1 to another single digit 0 or 1; the power and speed of the machine depends on the possibility of doing these elementary operations very fast, perhaps at a rate of the order of a million a second, and in a way which can easily be organised and controlled.

Multiplication can either be done by successive addition or by the use of a multiplication table; again the distinction is only significant in a decimal machine.

The treatment of signs and of the position of the decimal or binary point I must leave with the bare mention that they have to be considered. There are several possibilities, but it would take too long to discuss them.*

6. Form of instructions.

In Babbage's "Analytical Engine", instructions were represented in a quite different way from numbers, and stored in a different kind of storage system, numbers being represented by the positions of a set of counting wheels and instructions by holes punched in a set of cards. A similar difference between the representation and storage of numbers and of instructions is seen in the earlier machines actually constructed, two of which are referred to in Section 7. But in more recent machines, including most if not all those completed in the past year or still under construction, instructions are coded and stored in such a way that, in the machine, they have the same form as numbers, the difference between them being in the use made of them. For example, in a serial machine in which they are represented by trains of electrical pulses, the pulse trains representing instructions are transferred from the store to the control unit, whereas those representing numbers are transferred between the store and the arithmetical unit, these transfers themselves being controlled by the control unit.

Whatever the physical form of the store, it must consist of a number of identifiable storage locations, and it is convenient to think of these as numbered. The number which is the label of any storage location is usually called its "address", or the "address" of its content. $C(n)$ will be written for "the content of storage location n".

There are two main forms for instructions, which can be illustrated by an example. Suppose we want to form the sum of the contents of storage

* For a short discussion see reference 10, p. 64.

locations n_1 and n_2 and put the result in location n_3. This could be done by a single instruction which could be written shortly :

$$C(n_1) + C(n_2) \text{ to } n_3. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Alternatively, if the arithmetical unit is of a kind which has a component, usually called an " accumulator " (which will be shortened to " acc "), which accumulates the sum of numbers added into it until it is cleared, the required operation could be done by three instructions :

$$\left.\begin{array}{l} C(n_1) \quad \text{to Acc,} \\ C(n_2) \quad \text{to Acc,} \\ C(\text{Acc}) \text{ to } n_3. \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Each instruction of the form (1) specifies three addresses in the store, whereas each instruction of the form (2) specifies a single address ; these forms of instruction are consequently known as the " three-address " and " one-address " forms respectively.

As well as specifying the individual operations, it is necessary to specify the sequence in which they are to be carried out. In a machine in which instructions are represented in the same form as numbers, this can be done in two ways. One is to include in each instruction the address from which the *next* instruction is to be taken ; with a three-address specification of the operation to be carried out, this gives a four-address form of instruction. Another way is normally to store instructions at addresses numbered serially in the same order as the time-sequence in which they are to be carried out. The address of the instruction currently being carried out is recorded in a register whose content is normally increased by unity on the completion of this instruction, and the content of this register is used to control the selection of the next instruction. A special kind of instruction is then necessary in order to depart from the serial order in which instructions are located in the store.

The control system of a machine will depend on the standard form adopted for instructions and the means adopted for selecting the next instruction, and once the machine is built, the instructions must conform to the type with which the control system has been designed to deal.

7. The Harvard Mark I Calculator and the ENIAC.

The first general-purpose automatic digital machine actually constructed was the " Harvard Mark I Calculator " ([1,11]) developed by Professor H. H. Aiken of Harvard and the I.B.M. Company, and now installed at the Computation Laboratory at Harvard University. This machine, which has been hailed as " Babbage's dream come true " ([7]), was projected before the war, but only completed during it.

It is a parallel decimal machine. It has a number of mechanical counters serving both as adding units and storage registers ; these are driven from a continuously-rotating shaft through electromagnetic clutches controlled by relays, which themselves are controlled by circuits made through contacts on other relays or counters, or through holes in punched cards or tape. Control of the sequence of operations is carred out by means of a paper tape, on which instructions, in coded form, are punched in successive rows. Circuits made through the punched holes operate relays which cause the operation specified by the punched holes to be carried out. Fig. 3 shows a general view of this machine and Fig. 4 a close-up view of the sequence control unit, with a sequence-control tape in position.

Use of electronic counting and switching circuits greatly increases the

speed at which the operations of a machine can be carried out. The first automatic digital machine using electronic circuits was the ENIAC ([8,9]) of Dr. J. W. Mauchly and Dr. J. P. Eckert. In this machine, electrical pulses produced at a frequency of 200,000 per second are routed to electronic counting circuits by electronic gate circuits. It is a parallel decimal machine, and consists of a number of units connected by three sets of electrical transmission lines, one supplying a standard pattern of pulses to all units, one (" digit lines ") for transferring pulses representing numbers between them, and another (" program lines ") for transferring control pulses which set the units into operation and so control the sequence of operations carried out by the machine.

Numbers known before the calculation is started can be supplied to the machine through banks of hand-set switches (used for tabular data) or through punched cards from which they are transferred to relay registers ; intermediate results are stored on electronic counting circuits which also serve as adding units. The storage for instructions consists of the connections between the units and the digit and program lines, and the settings of various control switches on the various units, all of which have to be made by hand before the machine is ready to start the calculation.

Fig. 5 shows a general view of the ENIAC, and Fig. 6 shows a close-up view of two units with their connection to the digit lines (centre) and program lines (below).

For fuller accounts of both these machines the reader should refer to the references already given.

8. *Recent developments.*

Machines developed recently differ from those mentioned briefly in the previous section in two important features.

First, they have a much greater storage capacity, while at the same time being mechanically and electrically simpler. This is due to the development of different forms of storage giving large capacity without a large amount of equipment. There are three forms of storage which have been developed for this purpose : (i) acoustic delay units, in which information is stored in the form of trains of acoustic pulses in a column of mercury contained between two piezo-electric crystals by which electrical pulses can be changed into acoustic pulses and *vice-versa* ([21]) ; (ii) magnetic material on which information is stored in the form of the magnetic state of the material ([5]) ; (iii) an insulating screen on which information is stored in the form of an electrical charge distribution. The first is used in several machines ; the second is used as the main store in the Harvard Mark III machine recently completed, and as an auxiliary store in several other machines. A form of the third has been developed by Professor Williams at the University of Manchester ([22]), where a machine using this form of storage has been built.([14])

The second feature of most recent machines has already been mentioned in Section 6, namely, the use of the same form, within the machine, for instructions as for numbers. This has the important consequence that instructions themselves can be treated as numbers, and transferred, built up or modified by using the facilities of the arithmetical unit. If the storage locations are distinguished by numbers, an instruction will usually contain at least one number, an address to which that instruction refers, so that it is quite a straightforward use of the arithmetical unit to take an instruction and, for example, add unity to an address specified in it. This freedom to modify instructions in the course of the calculation, and moreover to modify them, if required, in a way depending on results of the calculation itself, makes the process of drawing up the schedule of instructions a very flexible one, and

easily applicable to calculations involving much discrimination and selection between alternative procedures.

9. *The EDSAC.*

Rather than trying to summarise the features of a number of machines, which would involve saying little about any one of them, I will restrict myself to the one with which I am most familiar, the EDSAC (Electronic Delay-Storage Automatic Calculator) at the Mathematical Laboratory of Cambridge University ([19,20]).

This is a serial binary machine with a storage system consisting of ultrasonic acoustic delay units. It uses a one-address form of instructions, and serial storage for them. Its general organisation is shown, somewhat simplified, in Fig. 7 ; crosses indicate gates which are operated by the control unit and control the transfers of pulse trains between the different units of the machine.

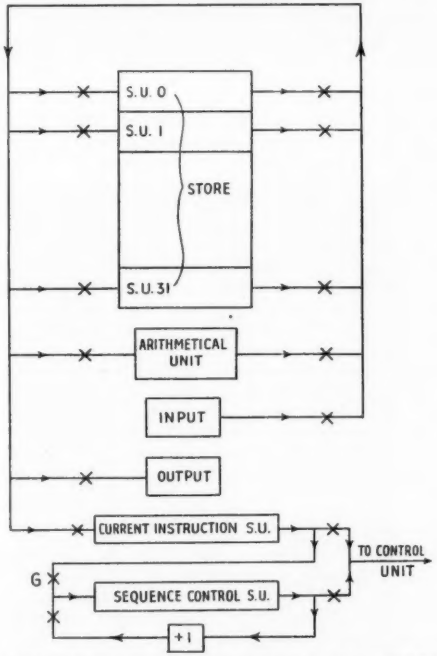

FIG. 7. Simplified form of general organisation of the EDSAC.
Crosses indicate gates operated by the control unit.
S.U. = Storage unit.
S.U. 0 to S.U. 31, storage units of main store, for numbers and instructions.
" Sequence control S.U." : used for the *address* of the current instruction ;
" Current instruction S.U. " : used for this instruction itself.

There is one storage unit (marked " current instruction S.U." in Fig. 7) for registering the current instruction, and another (marked " sequence control S.U.") for registering the address from which the current instruction is

taken. The content of the latter is normally increased by unity for every instruction carried out, but it is possible to transfer to it the address specified in the current instruction itself, and so to depart from the sequence in which the instructions are stored. This is done by opening the gate marked G in Fig. 7.

The machine has two stages of operation which are carried out alternatively. In the first an instruction is selected according to the number standing in the sequence control unit, and placed in the current instruction register. In the second this instruction is transferred to the control unit, where it is set up on a static register; the content of this register then operates selectors controlling the gates which determine the operation carried out.

Numbers and instructions are input to the machine in coded form on teleprinter tape and results are output on a modified teleprinter typewriter. Numbers are input and output in decimal form, the decimal-binary conversion on input and binary-decimal conversion on output being carried out by the machine in accordance with instructions for these processes.

A general view of the EDSAC is shown in Fig. 8. The tape reader and teleprinter typewriter are on the table to the right in this figure.

10. *Programming and coding.*

This and the following section are concerned with the organisation of a calculation for an automatic digital calculating machine. This has two stages. The first, sometimes distinguished as "programming", consists in breaking down the calculation to a sequence of elementary operations which the machine can carry out, such as addition, multiplication, transfer, and selection of the next instruction. A similar process is required in a hand computation; before we can do any calculation we must decide just how we are going to set about doing it. And in outline this stage is much the same whatever the machine we are going to use for the calculation, though in details it depends on the machine, since not all machines have the same basic set of elementary operations; in some, for example, $|x|$ can be found from x by a single operation, in others it has to be programmed.

The second stage, distinguished as "coding", consists in drawing up the schedule of operating instructions in the form in which they are to be furnished to the machine. This does depend on the particular machine; not on the physical form of the store or on the way in which numbers are represented and operated on within the machine, but mainly on two features, namely, the standard form adopted for instructions and the means of selecting the next instruction.

11. *Example of coding*

The different forms of instruction, and different ways of selecting the next instruction, make it difficult to generalise, or to illustrate the different possibilities in a short space. I will, therefore, again restrict myself to the machine with which I happen to be most familiar. The main features of it which we need are that it uses a one-address form of instruction with serial storage, and that the system of storage is such that the content of a storage location remains unchanged until another number or instruction is transferred to it. The following are the shortened forms of the instructions which are necessary for the example which I will consider here (for the complete set of elementary operations, and the form of the instructions for them, see reference 18).

$A\ n$ Add $C(n)$ into the accumulator.
$S\ n$ Add $-C(n)$ into the accumulator.
$T\ n$ Transfer $C(\text{Acc})$ to storage location n and clear the accumulator.

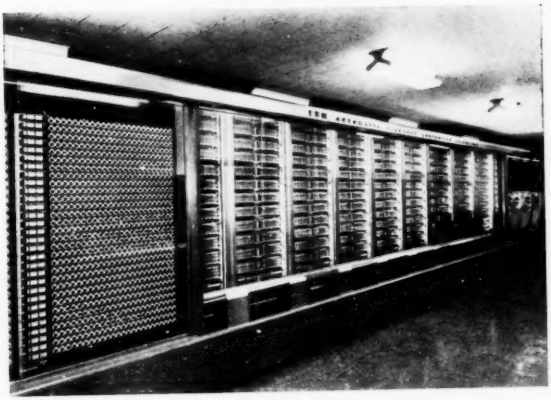

FIG. 3.

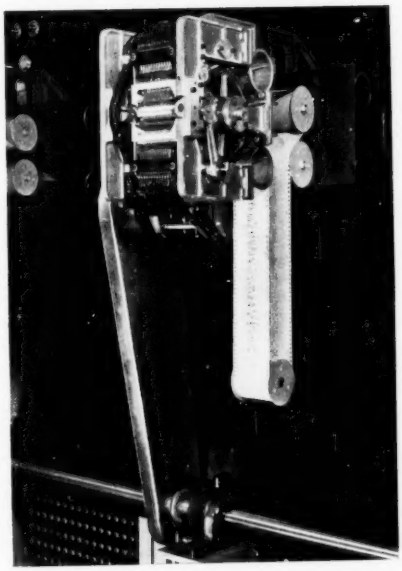

FIG. 4.

FIG. 5.

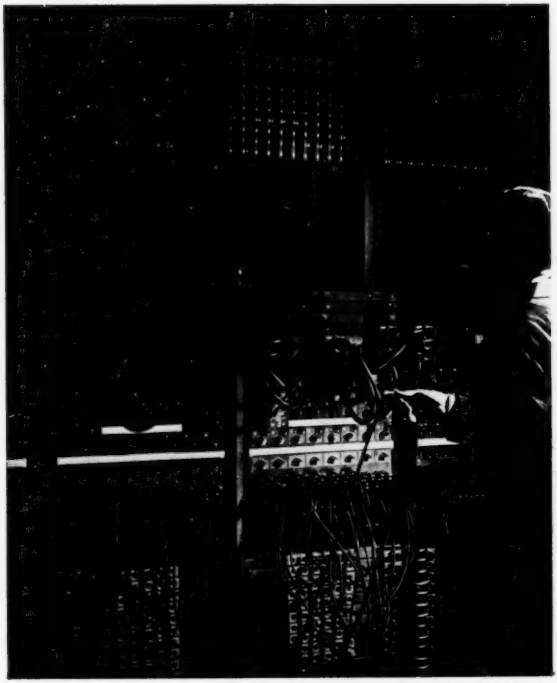

FIG. 6.

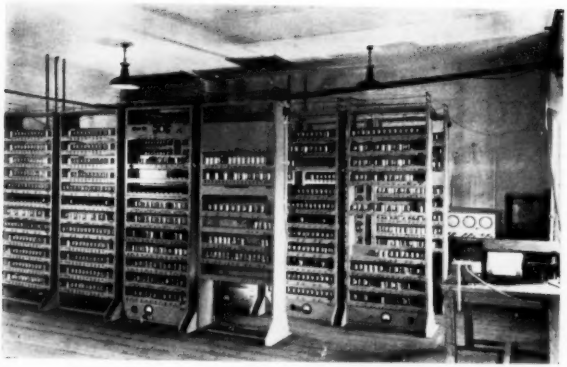

FIG. 8.

U n Transfer C(Acc) to storage location n and retain it in the accumulator.

E n Examine the content of the accumulator; if zero or positive, take $C(n)$ as next instruction; if negative, proceed serially (that is, if this instruction has been taken from address m, take $C(m+1)$ as the next instruction).

Consider the following calculation (thought of as a part of a larger calculation): Add the contents of storage locations 100 to 140 and place the result in location 170, or symbolically:

$$\sum_{j=0}^{40} C(100+j) \text{ to } 170. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The simplest and most straightforward way of programming and coding this would be by the sequence of instruction *A* 100, *A* 101, *A* 102, ..., *A* 140, *T* 170. This supposes the accumulator to be initially clear as a result of the previous instruction; if not, it is first cleared by the instruction *T* 0; it can only be cleared by *T*-instruction, and storage location 0 is conventionally used as a "rubbish bin" to which contents of the accumulator which are no longer wanted can be transferred in order to clear it.

This sequence of instructions is extravagant of storage space, and the calculation can be done much more economically as follows. We take the group of instructions *A* 170, *A n*, *T* 170, which result in $C(170)$ being increased by $C(n)$, and use this group of instructions over and over again, with n having in turn the values 100, 101, 102, ..., 140. This means that we must increase the value of n, the address specified in the instruction *A n*, by unity each time this instruction is used.

This can be done by taking the instruction *A n* itself into the arithmetical unit and adding to it unity in the digital position corresponding to a unit in n. The number consisting of unity in this digital position is kept in storage location 2 for just this purpose.

These instructions must be preceded by preliminary instructions to clear the accumulator and location 170. Thus so far we have the following instructions, at addresses starting from, say, m.

Storage Location	*Instruction*	*Notes*
m	*T* 0	To clear accumulator. } Preliminary preparation.
$m+1$	*T* 170	To clear 170. }
$m+2$	*A* 170	The addition process; the instruction in $(m+3)$ becomes successively *A* 100, *A* 101, *A* 102, ..., *A* 140 as result of instructions in $(m+5)$ to $(m+7)$.
$m+3$	*A* 100	
$m+4$	*T* 170	
$m+5$	*A* $(m+3)$	Increases by unity the address specified in instruction in $(m+3)$.
$m+6$	*A* 2	
$m+7$	T $(m+3)$	

Each time this set of operations is repeated, the address specified in the instruction in location $(m+3)$ is increased by unity. Before repeating it again, the machine must test that the addition is not yet complete. If the last number added was $C(n)$, then after the instruction in address $(m+6)$ has been carried out, the content of the accumulator is $A\ (n+1)$. If the instruction in $(m+7)$ is replaced by $U\ (m+3)$, then $A\ (n+1)$ will be retained in the accumulator after being transferred to $(m+3)$. If then *A* 141 is subtracted from the accumulator, the result is $(n-140)$, which is negative if the addition process is not complete and zero if it is complete. It will be found most convenient to have *A* 141 in location $(m+12)$, and the next instruction,

after the addition is completed, in $(m+13)$. Then the following instructions serve to switch the calculation from a repetition of the addition process to the next stage of the calculation, once the addition is complete :

Storage Location	*Content*	*Notes*
$m+7$	$U\ (m+3)$	In place of $T(m+3)$ of previous schedule.
$m+8$	$S\ (m+12)$	
$m+9$	$E\ (m+13)$	If the addition is complete, $C(\text{Acc})=0$ and next instruction is taken from $(m+13)$; if it is incomplete, $C(\text{Acc})$ is negative, and the next instruction is taken from $(m+10)$.
$m+10$	$T\ \ 0$	Clears the accumulator.
$m+11$	$E\ (m+2)$	This instruction is only reached if the addition is incomplete ; $C(\text{Acc})=0$ as the result of the previous instruction, so the next instruction is taken from $(m+2)$ to repeat the addition process ; this jump back to the instruction in $(m+2)$ is made with the accumulator clear.
$m+12$	$A\ 141$	
$m+13$		The next instruction after the addition is complete.

The instruction A 141 in location $(m+12)$ here is never actually acted on ; it is only used for the purpose of comparison with the instruction $A\ n$ manufactured by the machine, to test whether the addition is complete or not.

Thus with this method of programming and coding the addition, only thirteen instructions are necessary to carry out the calculation (3) ; and this is so for any similar addition, *however many numbers have to be added*. This illustrates that the number of instructions required to carry out a calculation does not necessarily increase with the total number of operations to be carried out.

In using these instructions, the number 100 which is the address from which the first addend is to be taken has been lost, since the content of $(m+3)$ has been changed in the course of the calculation. If these instructions form part of a bigger group which may have to be repeated in the course of a single calculation, this instruction A 100 must be replaced in $(m+3)$; or better, it can be stored somewhere else and transferred to the appropriate address by a preliminary operation before the addition process is started.

It will be seen that some of the instructions in the above schedule refer to storage locations which involve the value m, that is to say they depend on the position in the store in which these instructions themselves are placed, whereas others are independent of this position. However, in drawing up the schedule of instructions, and in punching them on the tape, it is convenient to express them in a form which does not explicitly involve the value of m. This is done by including in each instruction, as punched on the tape, a code letter ; F for one in which the number is to be read as it stands as the address to which it refers, and θ for one in which the number punched has to have the value of m added to it in order to give the address to which it refers.

Thus these instructions take the form :

Storage Location	*Instruction*	*Entry on tape*
		G K
m	$T\ \ 0$	$T\ \ 0\ F$
$m+1$	$T\ 170$	$T\ 170\ F$
$m+2$	$A\ 170$	$A\ 170\ F$
$m+3$	$A\ 100$	$A\ 100\ F$

$m+4$	T 170	T 170 F
$m+5$	A $m+3$	A 3 θ
$m+6$	A 2	A 2 F
$m+7$	U $m+3$	U 3 θ
$m+8$	S $m+12$	S 12 θ
$m+9$	E $m+13$	E 13 θ
$m+10$	T 0	T 0 F
$m+11$	E $m+3$	E 3 θ
$m+12$	A 141	A 141 F
$m+13$		Next instruction after addition complete

The code letters GK, F and θ do not appear in the instructions in the form they take in the store; they are instructions to the machine regarding the interpretations of the punchings on the tape and the formation of instructions from them. The entry GK on the tape is a coded instruction to the machine that in the process of reading the entries on the tape and forming from them the instructions to be placed in the store, the address of the storage location into which the next instruction is placed is to be recorded so as to be available for the process of forming the addresses in those of the subsequent instructions which, as punched on the tape, are distinguished by the code letter θ. The code letters F and θ serve to identify the end of the number in the instruction which specifies the address to which it refers.

As thus coded on the tape, the instructions have the same form wherever they are to be placed in the store, and once drawn up, punched, and verified they can be copied as they stand whenever the calculation which they carry out is required. There is auxiliary equipment for carrying out this copying mechanically and for checking the copy.

There is an important extension of this use of code letters to indicate operations to be carried out on entries as punched on the tape in forming instructions or numbers to be placed in the store. This enables the *same* set of punched instructions to be used for *any* calculation of the type

$$\sum_{j=a}^{b} C(j) \text{ to } c,$$

whatever the addresses a, b, and c occurring in a particular application. The values to be assigned to the parameters a, b, c must, of course, be specified to the machine in each particular case; this is done by separate entries on the tape, preceding the instructions themselves.

The operation of the machine, including the reading and interpretation of the entries on the input tape, can only be controlled by instructions taken from the store, and when the machine is first switched on there are no instructions in the store. A standard set of initial instructions is therefore wired permanently on a bank of uniselectors (rotary multi-way multi-pole switches). When the starting button is pressed, these initial instructions are transferred to the store, and they are designed so as to enable the machine then to read and interpret the entries on the input tape.*

One comment in conclusion. These machines have been called "electronic brains", which carries the suggestion that they can "think for themselves", which they cannot do: they can only carry out, quite blindly, the sequence of instructions which has been thought out for them. As Byron's daughter, Lady Lovelace, said ([15]) of Babbage's Analytical Engine over 100 years ago,

* For details of these initial instructions and the relations between them and the coding of instructions on the tape, see reference 17.

" the machine has no pretensions to *originate* anything ; it can only do *what we know how to order it* to perform ". And this is as true now, of the machines that have so far been built, as it was when it was written. D. R. H.

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Note.—This bibliography is far from complete, and includes mainly those references which are relatively easily accessible to British readers. A complete list of references would include over 100 items.

GLEANINGS FAR AND NEAR.

1645. Mr. Churchill can now wear 23 medals plus the O.M. and the C.H. (which equals 25).—*Sunday Dispatch*, May 11, 1947. [Per Mr. J. T. Pye.]

GEOMETRY AND GIRARD DESARGUES.

By B. A. Swinden.

It is a far cry from the Conics of Apollonius to the theory of linear spaces and matrix algebra and many men have helped to build the road between. This article is about one of the builders whose work for a time was discarded and whose name was almost forgotten. But his name finds frequent mention now, when the latter aspect of geometry bears so little resemblance to the former, and metrical geometry has given place to projective geometry with its non-metrical homogeneous coordinates, which make it possible to publish a book on coordinate geometry which does not contain a single figure.

It was only about a century ago that projective geometry began to develop as an independent branch of mathematics. Before that period its basis was almost entirely metrical. Such ideas as harmonic section, foreshadowed by Apollonius himself (c. 260 B.C.–200 B.C.) in his fourth book, were defined essentially in terms of distance. In the *Συναγωγή* of Pappus (end of 3rd century A.D.) there is the important theorem that if $A_1A_2A_3$, $B_1B_2B_3$ be two triads of points lying respectively on each of two straight lines, then the cross-meets such as (A_2B_3, A_3B_2) are collinear, a theorem later generalised by B. Pascal (1623–1662), who replaced the pair of lines by any conic. With Pappus' Theorem and the fundamental propositions of incidence in a plane, *viz.* that given two distinct coplanar lines there is a unique meet, and given two distinct points there is a unique join, we can, without any use of the idea of distance, assign to each point of the plane two (or three) coordinates which are elements of a collection satisfying the ordinary formal laws of algebraic addition and multiplication and are such that the coordinates of any point on the line joining two distinct points P and Q are linear combinations of the coordinates of P and Q, *i.e.* a straight line has a linear equation.

Pappus, too, discussed the problem *ad tres et quattuor lineas*, wherein a conic is found to be the locus resulting from the motion of a point the product of whose distances from two straight lines is equal to its distance from a third, or to the product of its distances from a third and fourth lines : the ancients had proved this for the case of three lines, but Pappus, while stating the result for four lines, gave no proof in that instance. Newton (1642–1727) gave a purely geometrical solution, in which Descartes had failed, though his method of coordinates enabled him to prove that the locus was a curve of the second order.

G. Desargues (1593–1661), who introduced the conceptions of points at infinity, conical and cylindrical projection, involution, pole and polar and the germ of the notion of cross-ratio, and R. Descartes (1596–1650) with his development of analytical methods, were contemporaries who were in constant communication and whose ideas interacted. For reasons which I suggest later Descartes' work gained considerable celebrity while that of Desargues was neglected and would have fallen into complete oblivion but for a brief mention in Blaise Pascal's *Essay on Conics* and the researches of M. Chasles.

In the early nineteenth century interest in Projective Geometry revived with L. N. M. Carnot (1753–1823), J. V. Poncelet (1788–1867), who systematically used " imaginary " points, and J. Steiner (1796–1863), but the basis was still essentially metrical. Then in 1847, K. G. C. von Staudt (1797–1867) in his *Geometrie der Lage* expounded the view that Projective Geometry should be built on an abstract basis with no reference to metrical ideas, and F. C. Klein (1849–1925) pointed out that " Projective Geometry is the study of properties invariant under a certain group of transformations, which are in fact represented analytically by linear homogeneous transformations of coordinates ".*

* J. A. Todd, *Projective and Analytical Geometry*, 1947.

Bobillier and J. Plücker (1801–1868) could not have foreseen what would spring from their " methods of abridged notation " which were the genesis of so-called " trilinear " coordinates.

Girard Desargues is known to schoolboys by reason of two theorems with which his name is connected. In the ordinary Histories of Mathematics he usually receives only a few lines commemorating his theorem on the involution of the four-point conic ; certainly Rouse Ball* gives him credit for the invention of Projective Geometry, but he places him, wrongly I think, at the end of an epoch. Of his work on Conics I shall speak later, but his theorem of homologous triangles may be mentioned now. Pappus' Theorem together with the propositions of incidence in a plane, as I have said above, enables us to establish a one-to-one correspondence between each point of the plane and a set of homogeneous coordinates which are elements of a field ; if instead of Pappus' Theorem we use Desargues' Triangle Theorem, our coordinates still satisfy the basic laws of algebra except the commutative law of multiplication. In the special case when the two triads are in perspective Pappus' Theorem can be deduced from Desargues', but not otherwise. In projective space of three or more dimensions Desargues' Theorem follows from the basic propositions of incidence and need not be† adjoined to them.

Desargues was born at Lyons in 1593 ; his father was a man of good family, apparently a notary, who had some property near Condrieu, to which Girard is said to have retired at the end of his life. We know nothing of his early life or education, nor of how he became interested in geometry ; presumably he had read Euclid, Apollonius and Ptolemy, since he quoted them in his work : indeed much of our knowledge of him comes only from sources such as Baillet's *Life of Descartes* (1691) and from the *Letters of Descartes* published by Cousin in Paris, 1824–6.‡ Desargues was certainly in Paris in 1626 when Descartes went there to settle ; their community of age and interest brought them together and they formed a lasting friendship. Both appear to have been interested in applying mathematics to the easement of labour. It seems certain that Desargues was not unskilled in engineering and architecture § since Cardinal Richelieu employed him in this capacity at the siege of la Rochelle in 1628. At the end of the war he returned to Paris where he devoted himself to the study of geometry and its applications to architecture. He joined the circle which met weekly at le Pailleur's house to discuss mathematics and natural philosophy : these meetings led to the formation of the French Academy in 1666. There he met Gassendi, Bouillaud, Pascal the elder, Roberval, Mersenne,‖ Mydorge : it was in 1637 that Blaise Pascal, then aged

* *Short History of Mathematics*, sixth edition, p. 257.

† For further discussion see G. de B. Robinson, *The Foundations of Geometry*, Toronto, 1946. Hilbert, *Grundlagen der Geometrie*, seventh edition, pp. 85 *et seq.* shows that D.'s theorem cannot be proved in a plane without using a third dimension or some congruence axiom, and gives an example of a two dimensional space in which the theorem is not true.

‡ M. Poudra (*Oeuvres de Desargues, Paris*, 1864) collected practically every literary reference which would help to build up a picture of the history and character of D., and I acknowledge my debt to him.

§ The archives of Lyons contain some correspondence which passed in 1646 between the municipal authority and " the celebrated architect Desargues, a son of the borough " relating to plans for a new town hall.

‖ Mersenne was very much concerned with mathematicians and exponents of other branches of learning, and seems to have acted as a " clearing-house " of knowledge ; results were frequently communicated to him before publication and letters from one scientist to another were passed through him ; *e.g.* Desargues and Beaugrand, the King's Secretary, who were not on speaking terms, communicated through the inter-

14, was admitted to this company, at the time when Desargues was engaged on his work on Conics, a point of some importance in view of later events.

Desargues' interests seem to have been very wide. He wrote, we know, on pure geometry, mechanics, architecture, gnomonics and perspective : of his work on algebra, which Descartes asked to see,* we have no knowledge, nor can we say to what extent he studied metaphysics, though we know that Descartes preferred his judgment on such matters " to that of three theologians ".†

In order correctly to appreciate Desargues' work it is necessary to consider the background against which he wrote. After a final century of infertile existence Alexandria fell in A.D. 641 and the long history of Greek mathematics closed. Thereafter in Europe mathematics stagnated for some centuries and geometry practically disappeared. It is true that in the Middle Ages the monastic schools and some learned clerks, particularly those connected with building operations, kept alive a faint memory of an almost legendary figure called Euclid, who was the " inventor " of geometry and who was mixed up in some curious way with Egypt and perhaps with Abraham! ‡ but speaking generally geometry was dead : its knowledge was confined to certain practical rules of measurement and results such as Pythagoras' theorem, which were almost trade secrets of the hierarchy of professional builders. During the twelfth century copies of Arabian and Greek textbooks were introduced into western Europe through the Moors of Spain, but they remained very rare : the fall of Constantinople, too, sent a wave of refugees into Italy, bringing with them some of the traditions of Greek science. Then the invention of printing in the middle of the fifteenth century, allowing the wider dissemination of the results obtained by the Arabs and the Greeks, heralded the Renaissance. During the next two centuries mathematicians were largely concerned with the development of algebra—first syncopated, then symbolic—and trigonometry (it may be noted that Desargues' work on Conics is marked by a complete absence of algebraic symbols) : only about the beginning of the seventeenth century do we find a reviving interest in mechanics and geometry. The latter revival was perhaps due to J. Kepler (1571–1630) who, in a short note on Conics in his *Paralipomena* (1604), asserted the principle of continuity, introduced the name " focus " and suggested the use of the eccentric angle in dealing with the ellipse. §

Thus at this period there was a marked resurgence of mathematical progress but interest was generally concentrated on the development of analysis. Vieta's (1540–1603) introduction of symbolic algebra, brought into general use by the writings of Harriot (1560–1621) and Descartes, Cavalieri's (1598–1647) use of indivisibles,‖ Roberval's (1602–75) discussion of the nature of tangents to curves, Fermat's (1601–65) work on infinitesimals and the use of analysis in geometry, and above all Descartes' introduction of analytical geometry, were all symptomatic of the trend of the times. True, in 1522 J. Werner of Nuremberg published a small pamphlet on Conics, and F. Maurolicus (1494–1575) of Messina translated various Latin and Greek texts and discussed the conics, as

mediary of Mersenne. It may be that the *Letters of Mersenne*, published by Mme. Tannery during the last fifteen years, would throw still more light on Desargues, but I have not seen the relevant volumes.

* Cousin, vol. 8, p. 493, 28th February, 1641.

† Cousin, vol. 8. p. 433, 31st December, 1640.

‡ *British Museum, Add. MS.*, 23,198 vv. 435–538.

§ Ball, l.c. p. 256.

‖ Archimedes rejected a similar method as not rigorous, v. *The " Method " of Archimedes*, edition T. L. Heath, 1912, p. 13.

Werner had done, as sections of a cone, but the general feeling, expressed by Descartes in one of his letters to Mersenne,* was that pure geometry had no more to say,† and this impression persisted for well over a century.

One other point should be mentioned. Even at this time a writer frequently published his work merely by printing a few broadsheets in microscopic characters and circulating them among his immediate contacts : this method was certainly adopted by Desargues, which may account for the rarity of his written remains. The procedure did not make for rapid diffusion of ideas and it lent itself to the unscrupulous appropriation of other men's discoveries, leading to acrimonious disputes about priority of authorship.

Such broadsheet publications were the following :

1. *Méthode universelle de mettre en perspective les objets donnés réellement, ou en devis, avec leur proportions, mesures, esloignemens, sans employer aucun point qui soit hors du champ de l'ouvrage, par G. D. L., Paris* 1636 *avec privilége* (*de* 1630).

This was reproduced at the end of Bosse's *Perspective* of 1648, together with a note containing a number of Desargues' short memoirs designed to elucidate the main work, *viz.*

2. (*a*) *Proposition fondamentale de la pratique de la perspective.*
   (*b*) *Autre fondement encore du trait de la perspective, ensemble du fort et du faible de ses touches ou couleurs.*
   (*c*) *Proposition géométrique* [this is Desargues' Triangle Theorem].
   (*d*) Two other geometric propositions.

3. *Perspective adressée aux théoricians :* reproduced at the end of Bosse's *Perspective* of 1648 and said by Curabelle to have been printed in 1643.

4. *Brouillon project d'exemple d'une manière universelle du S.G.D.L. touchant la practique du trait à preuves pour la coupe des pierres en l'architecture, et de l'esclaircissement d'une manière de réduire au petit pied en perspective comme en géometral et de tracer tous quadrans plats d'heures égales au soleil. Paris en aout* 1640 *avec privilege.*

In this he speaks of Fermat's method of drawing tangents and his theory of maxima and minima and he credits Roberval with the discovery of the curve " traced by a point in the diameter of a circle which rolls on a straight line ". He also names as his own pupils Bosse the engraver, de la Hire the painter‡ and Hureau a master-mason, who, he says, understood his method of perspective in less than two hours.

5. *Manière universelle de poser le style aux rayons du soleil en quelque endroit possible avec la règle, l'esquerre et le plomb.* 1640.

M. Poudra reconstructed this work from the criticism *Avis charitables* mentioned below.

6. *Atteinte aux évènemens des contrarietez d'entre les actions des puissances ou forces.*

This was a short memoir which is lost except for a few lines quoted by Beaugrand. It seems to have been appended to the Treatise on Conics, and makes use of the word " involution " defined for the first time in that work. In effect it states that the centre of gravity of a sphere lies in the diameter common to the earth and sphere at the point conjugate to the earth's centre.

* Cousin, vol. 8, p. 88.

† J. L. Coolidge, *History of Geometrical Methods*, 1947, pp. 104–5, expresses the same view with regard to the present situation in synthetic projective geometry.

‡ Father of Philippe de la Hire, the geometer (1640–1719).

Descartes, in a letter to Mersenne dated 15th November, 1638, commenting on this and partially correcting it, gave an incorrect result but was unable completely to solve the problem.

Desargues' most important publication was his Treatise on Conics, entitled *Brouillon project d'une atteinte aux évènemens des rencontres d'un cone avec un plan, par le sieur G. Desargues Lionais, Paris* 1639.*

The treatise is not easy to read, partly because, as Desargues says in his title, it is no more than a rough sketch or outline, partly because he really lets himself go in the invention of new and unusual terms. He opens with an expansion of Kepler's principle of continuity : thus parallel lines meet in a unique point at infinity, the (infinite) points at opposite " ends " of a line being regarded as coincident ; parallel planes meet in a unique line at infinity ; a straight line is the special case of a circle whose centre is at infinity. Then he continues with a large number of definitions leading up to the theory of involution of six points ; various special cases are treated, particularly that in which certain coincidences reduce the six points to four, giving a harmonic range. The theory of projectivities on a line revolves round the double points and Desargues was quick to notice this : he establishes that the characteristic cross-ratio of an involution is -1, and obtains most of the relations between segments of a harmonic range which now appear in school textbooks. He defines involution and harmonic pencils and deduces the important result that the characteristic property of an involution or harmonic range is projective. His main weapon in establishing these results is Menelaus' Theorem on the plane triangle cut by a transversal.

Coming now to the conic, Desargues defines pole and polar and remarks that the centre and diameters are special cases. He then obtains the theorem of the involution of the complete inscribed quadrangle by projection from the circle and goes on to the self-polar triangle, with more theory of poles and polars, the involution of conjugate points and an involution of points on a conic. In a general proposition he shows how the axes, conjugate diameters, tangents, asymptotes (*viz.* : tangents at infinity) and foci may be obtained by projection from a circle or from another conic. He also introduces the idea of a polar plane for a solid.

Between 1626 and 1630 Desargues gave some lectures in Paris which greatly impressed his colleagues : these were probably on perspective,† architecture and gnomonics. He himself says that he was impelled to study the geometrical basis of these subjects because having noticed that the master-craftsmen, architects, painters, masons and so on, had learnt only a number of empirical rules and obtained their results in a random and groping manner resulting in uncertainty and fatigue, he was inspired with a desire to lighten their labours by discovering shorter and more intelligible rules. The pamphlets 1, 4 and 5 clearly represent the issue of his researches.

It will be noticed that all these publications emanated between 1630 and 1643. Desargues was highly appreciated by his mathematical contemporaries as is obvious from the letters of Descartes and that passage in the younger Pascal's *Essay on Conics* in which he said : " We shall also prove the following property due to M. Desargues of Lyons, one of the great intellects of this time and amongst those most expert in mathematics, particularly in conics ; his writings on this subject, though not numerous, have given ample testimony of

* D.'s critics, Curabelle and others, spoke of a work on Conics called *Leçons de ténèbres* of which we have no trace. I suspect that this was merely an alternative title to the *Brouillon-projet*.

† Coolidge, l.c. p. 109, calls Desargues " the bad boy of the whole story " of Perspective.

his ability to those who have been willing to take cognisance of it. I gladly acknowledge that I owe the little I have discovered in this subject to his writings and that I have tried to imitate, so far as I could, his method of treating all the conic sections generally without using the triangle through the axis. The remarkable proposition in question is etc.—" and he went on to enunciate the involution of the inscribed quadrangle. Descartes did not think that Pascal could be the sole author of this essay and attributed it to the teaching of Desargues,* while Leibnitz said : " I think M. Descartes was right in saying that young Pascal, aged only 16 when he composed his treatise on conics, had profited by the ideas of M. Desargues and it seems clear to me that Pascal acknowledged that himself."

But Desargues came up against professional jealousy an the part of his non-mathematical colleagues. In 1642, François l'Anglois *dit* Chartres published a book from the press of Melchior Tavernier entitled *Practical Perspective by a Parisian Jesuit.*† This book contained serious errors and Desargues criticised it severely in notices placarded on the walls of Paris and in a small pamphlet accusing the author of having clumsily plagiarised his own methods. The author and publisher, interested in the sale of their production, retaliated by printing several violent attacks on Desargues. They made a collection of these libels and of a number of adverse criticisms of Desargues' works which were current and published them under the ironical title *Advis charitables sur les divers œuvres et feuilles volantes du sieur Girard Desargues, Lyonnois. Paris* 1643. This collection contains items printed in 1640 and 1641 and different copies of it do not always contain the same items. In one of these variants appears the letter from Beaugrand,‡ the King's Secretary, dated 20th July, 1640, criticising and depreciating Desargues' " Conics " as no more than one of the lemmas of the seventh book of Pappus together with a corollary of the seventeenth theorem of Apollonius' third book : nearly the whole of the last half of the letter is concerned with Beaugrand's own discussion of the centroid of the triangle and the criticism of Desargues' note on mechanics already mentioned.

These attacks annoyed Desargues, who like Descartes had a difficult temperament and became restive under opposition ; he appears to have decided to write no more under his own name. One of his pupils was an engraver, Abraham Bosse, who though no mathematician had sufficient penetration to understand the practical application of Desargues' theories on perspective, and became his loyal supporter and friend. Desargues entrusted to Bosse the task of interpreting and working out in detail the general theory he had evolved, a task to which Bosse devoted himself with the utmost fidelity : indeed, in 1666 he published a full statement of the events which led him to resign from his post as Lecturer at the Royal Academy of Painting and Sculpture, from which it appears that he refused to obey the Governing Body's instruction to forswear the principles he had learnt from his master. Bosse's interpretations of Desargues' ideas were embodied in a number of books on Perspective, Gnomonics etc., dated from 1643 onwards : their style is very diffuse and they are consequently very difficult to follow, but three of them bear long forewords by Desargues stating that they are in complete conformity with his own methods and notions, and the " privileges " for these were granted on Desargues' application.

* Cousin, vol. 8, p. 214.

† Probably the pamphlet was the composition of Tavernier himself based on a MS. by the Friar Dubreuil.

‡ Beaugrand wrote a Commentary on the Cycloid, one or perhaps two books on " Geostatics " and a commentary on Vieta's principal work.

The critics were clearly of opinion that these three books, at least, were the work of Desargues himself, for in 1644 François l'Anglois published another attack entitled *Examen des œuvres du sieur Desargues par J. Curabelle*, in which Bosse's books were attributed to his master. The quarrel between Curabelle and Desargues, who was not a patient man, grew bitter and the latter offered to back his opinions with a stake of 100,000 livres : Curabelle accepted in the sum of 100 pistoles only and after long and acrimonious discussions on both sides negotiations broke down over the question of arbitrators. Curabelle gave his version of the story in a pamphlet called *Foiblesse pitoyable du sieur G. Desargues contre l'examen fait de ses œuvres par J. Curabelle.*

I have said that a community of temperament and interest produced a lasting bond between Desargues and Descartes. In 1628 the latter went to La Rochelle, curious to see the vast undertaking connected with the construction of the dyke " in the design of which his friend Desargues had some share ".* Even when Descartes withdrew to Holland in 1629 they maintained a regular communication and were in the habit of exchanging their views on mathematical themes, usually through the intermediary of Mersenne. Desargues used his influence with Richelieu to induce him to offer a pension to Descartes in order to persuade the latter to return to France. Descartes refused the offer but remained profoundly grateful. Desargues also supported Descartes in his quarrel with Bourdin and his discussion with Fermat. On the other hand Descartes took Desargues' part when Beaugrand was attacking the Treatise on Conics.

Though their methods of tackling the problem were different each had the same general outlook on mathematics and each in his own way carried his researches to a high degree of excellence. They were both highly sensitive to criticism and could not brook contradiction : each, aware of the value of his own concepts, was inclined to pay scant regard to the work of others.

Descartes died in Sweden in 1650 and Desargues felt his loss keenly. He withdrew to Lyons and appears to have carried out no more serious research though he continued to produce designs for some local building construction. He returned to Paris for a short visit in 1658 on the occasion of the marriage of a nephew who became his heir. He owned a country house at Condrieu to which he used to go to cultivate his garden. He died in 1661, a disappointed man. His will, dated 5th November, 1658, contained the following clause : " M. Desargues gives and bequeaths to his obliging and good friend M. Abraham Bosse, etcher, residing in Palace Yard, and in default of him to his heirs, the sum of 2000 livres payable in four instalments, etc."

Of Descartes, Pascal and Desargues the first two have lived while the last has been almost forgotten. Was Pascal so much greater a mathematician than Desargues, or can it be that his religious and philosophical *Pensées* have helped to keep his memory alive? He himself has acknowledged his debt to Desargues and is hardly likely to have been deceived in his estimate of the latter's ability. But Desargues laboured under great difficulties, some due to the trend of the times and some to his own temperament. Descartes' work was difficult to read, but it was in greater accord with current mathematical thought and it captured the imagination of contemporary scientists. Desargues had the odds against him in pursuing the course of pure geometry ; he lengthened those odds by the conciseness of his statements and the extraordinary nomenclature he adopted—he speaks of " trees ", " trunks," " knots," " branches " and all the rest of what Beaugrand called " rustic barbarisms "—and even Descartes was moved to chide him for departing from the language of Apollonius ; and the battle was finally lost through his quarrel with his critics, pursued with great

* v. Baillet's *Life of Descartes*, quoted by M. Poudra.

bitterness by him and his champion Bosse, whose break with the Royal Academy of Painting could not fail to have some effect on the newly formed French Academy. The quaintly named *Brouillon-projet* was forgotten and lost; all that remains is a copy made by Philippe de la Hire * together with a letter certifying its authenticity and a few notes made by him. These were found in a library in Paris by M. Chasles in 1845 and are now in the Library of the Academy: meanwhile de La Hire had been acclaimed as the author of the theory of poles and polars. With no knowledge of the actual work other than that to be derived from Desargues' critics, Chasles tried† to rehabilitate his reputation, but the pioneer was overshadowed by men like Monge, Brianchon and Poncelet who were founding the new school of descriptive geometry.

But Desargues' name is now being mentioned more frequently by geometers, and there is today a tendency to reduce previous estimates of what Pascal accomplished.‡

B. A. S.

* Ball, l.c. p. 317 says that P. de la Hire was Desargues' favourite pupil. I think this is wrong; anyhow Philippe was no more than ten years old when Desargues left Paris. Laurent, Philippe's father, was one of the Academicians who gave Bosse a declaration of thanks for his offer to publish his works in the name of the Academy.

† *Aperçu historique, etc.* 1837.

‡ Boutroux in Pascal's *Works* (Paris, 1923) vol. 1, pp. 245 *et seq.*

1646. On the subject of Education in its most extensive sense, an ancient writer "directs the aspirant after excellence to commence with the Science of Moral Culture; to proceed next to Logic; next to Mathematics; next to Physics; and lastly to Theology". Another writer on Education would place Mathematics before Logic, which (he remarks) "seems the preferable course: for by practising itself in the former, the mind becomes stored with distinctions; the faculties of constancy and firmness are established; and its rule is always to distinguish between cavilling and investigation—between *close reasoning* and *cross reasoning*; for the contrary of all which habits, those are for the most part noted, who apply themselves to Logic without studying in some department of Mathematics; taking noise and wrangling for proficiency, and thinking refutation accomplished by the instancing of a doubt. . . ." Preface to the 3rd edition (1850) of *Euclid's Elements of Geometry*, by Robert Potts, M.A., Trinity College.

[Pott's School Edition of Euclid, Bks. I–VI, XI, XII, contains Notes on each book—in those on Bks. II and V each proposition is demonstrated algebraically—some 390 questions on the propositions and about 680 geometrical exercises taken from Cambridge College and Senate House examinations. The earliest question, taken from the S.H. examination for degrees in 1801, is "Let AB and DC be two diameters of a given circle, at right angles to each other; AEB a circular arc described with radius DA or DB; prove that the area of the line $AEBC$ = area of triangle ADB"; this is given as an exercise on Bk. XII. (Potts considers the "method of exhaustions" strictly rigorous but too operose, and says "it has been replaced by the method of prime and ultimate ratios".) The Simson Line theorem was given as a Smith's Prize question in 1842, having previously appeared at Pembroke in 1830.] [Per Mr. B. A. Swinden.]

VARIETY IN TANGENTS AND NORMALS.

By C. O. Tuckey.

We seniors are apt to be told that, in our teaching of analytical geometry, we have been in the habit of giving too much attention to the intricacies of conic sections, and that work on other curves and their tangents and normals would have been more interesting and instructive. There is much force in this criticism, but when we come to examine the alternatives to the conics which are offered to us, they are usually cubic curves, probably given in terms of parameters, such as the " Folium of Descartes ". I should like to suggest that some better-known curves, notably the curve of sines and the logarithmic curve, will provide examples even more instructive than the " Folium ", and that these examples have at least the advantage of greater novelty.

Here are two results (α) and (β), which probably may be found buried in the archives of some mathematical society, but are not, I think, readily accessible to teachers in any of the better-known textbooks.

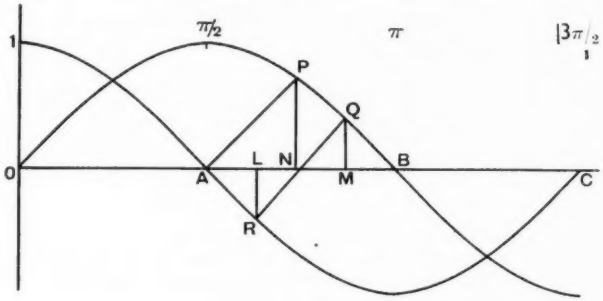


Fig. 1.

In Fig. 1 the curves $y = \sin x$ and $y = \cos x$ are drawn. Incidentally I suggest that, for classroom practice, it is so convenient to use the value $\pi = 3$, thus perhaps going back to the wisdom of Solomon, and it distorts the curves so little, that it may safely be recommended for rough drawings of these curves on squared paper.

(α) *If the normal at* $(\frac{1}{2}\pi, 0)$ *to* $y = \cos x$ *meets* $y = \sin x$ *at* P *and if the common normal to the two curves meets* $y = \sin x$ *at* Q, *then*

$$x_Q = \tfrac{1}{2}\pi + \tfrac{1}{2}x_P.$$

The equation to find the position of P, (θ), is $\theta - \sin\theta = \frac{1}{2}\pi$ (from $AN = PN$), which is readily solved approximately by the use of tables and gives $\theta \simeq 132° 21'$.

To deal with the common normal RQ, in order to avoid obtuse angles, it is convenient to shift the origin to A. The curves then change equations ; the upper one is $y = \cos x$ and the lower $y = -\sin x$. If AL is ϕ and AM is χ, then comparing gradients of the normals at R, Q we get $\sec\phi = \operatorname{cosec}\chi$, so that $\phi = \frac{1}{2}\pi - \chi$. The equations of the normals are, at R

$$y + \sin\phi = \sec\phi \,.\, (x - \phi) \quad \text{or} \quad y + \cos\chi = \operatorname{cosec}\chi \,.\, (x + \chi - \tfrac{1}{2}\pi)$$

and at Q

$$y - \cos\chi = \operatorname{cosec}\chi \,.\, (x - \chi).$$

On multiplying each equation by $2\sin\chi$ and comparing absolute terms, we get

$$\sin 2\chi + \pi - 2\chi = -\sin 2\chi + 2\chi,$$

which reduces to $$2\chi - \sin 2\chi = \tfrac{1}{2}\pi.$$

Hence $2\chi = \theta$, the angle represented by AN and determining P. Thus

$$2AM = ON\ ;$$

this is the result (α).

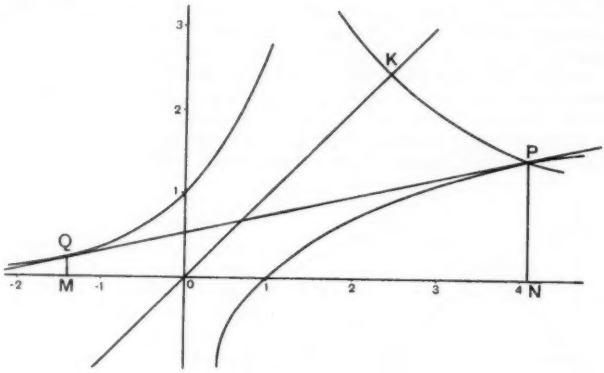


FIG 2.

In Fig. 2 the curves $y = \log x$ and $y = e^x$ are drawn.

(β) *If PQ is a common tangent touching $y = \log x$ at P, the position of P is found by the intersection of the curve with the rectangular hyperbola*

$$(x-1)(y-1) = 2.$$

The position of Q is found by making $OM = PN$.

Suppose ON to be x_1, so that PN is $\log x_1$, and OM to be of *length* x_2, so that Q is $(-x_2, \exp(-x_2))$. The gradients at P and Q are $1/x_1$ and $\exp(-x_2)$, so that $x_1 = \exp x_2$, and $x_2 = \log x_1$. Thus $OM = PN$, or $x_2 = y_1$.

Again, the two tangents are

$$\text{at } P \qquad y - y_1 = (x - x_1)/x_1,$$

$$\text{at } Q \quad y - \exp(-x_2) = \exp(-x_2)\,.\,(x + x_2)$$

$$\text{or} \qquad y - 1/x_1 = (x + y_1)/x_1.$$

Comparing absolute terms,

$$y_1/x_1 + 1/x_1 = y_1 - 1$$

or $$x_1 y_1 - x_1 - y_1 = 1,$$

that is $$(x_1 - 1)(y_1 - 1) = 2.$$

This shows that P is determined as stated in (β).

An attraction to the teacher is that such questions lead on to an indefinite number of others which may or may not be soluble in the same sort of way. It is only necessary to replace $y = e^x$ by $y = e^{\lambda x}$ and $y = \sin x$ by $y = \sin \lambda x$. For $y = e^{\lambda x}$ the corresponding rectangular hyperbola to that given above appears to be

$$(\lambda x - 1)(\lambda y - 1) = 2(1 + \log \lambda).$$

There is one question raised by the change to $y = \sin \lambda x$ which seems to be well worth considering.

(γ) *What is the least value of λ for which a normal to the curve $y = \sin \lambda x$ can meet the curve again?*

If a normal can meet the curve again, there must be two normals (or, in the limiting case, just one) which are also tangents. Now for the simple curve $y = \sin x$, the gradient of the tangent is a cosine and that of the normal a secant, so these cannot be the same. But for $y = \sin \lambda x$ this is not so. As λ increases (and the waves get rougher) the normal position becomes less upright and a normal can also be a tangent. In the limiting case, suppose that the unique normal to meet the curve again is the normal at Q and the tangent at P. Then

(I) *P must be the centre of curvature for Q.*

For if the centre of curvature were beyond P, a neighbouring normal to the right of Q would cut the curve again, while if the centre of curvature were between Q and P, a neighbouring normal to the left of Q would do so.

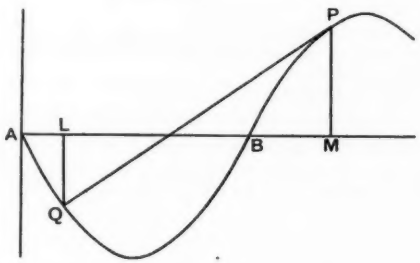


FIG. 3.

(II) It will be proved that $3AL + BM = AB$ in Fig. III, or $3\alpha + \beta = \pi/\lambda$. Note that for convenience the origin is taken at A, and so the curve is

$$y = -\sin \lambda x.$$

The gradient of the tangent at Q is $-\lambda \cos \lambda\alpha$ and that of the normal $(1/\lambda) \sec \lambda\alpha$. The gradient of the tangent at P is $-\lambda \cos (\pi + \lambda\beta) = \lambda \cos \lambda\beta$. These two are the same if

$$\cos \lambda\alpha \cos \lambda\beta = 1/\lambda^2. \quad \text{.....(i)}$$

Taking the normal at Q, namely,

$$y + \sin \lambda\alpha = (x - \alpha)/\lambda \cos \lambda\alpha$$

or

$$\lambda y \cos \lambda\alpha + \tfrac{1}{2}\lambda \sin 2\lambda\alpha = x - \alpha, \quad \text{.....(ii)}$$

and differentiating with respect to α, to get the intersection with the closely neighbouring normal, we have

$$-\lambda^2 y \sin \lambda\alpha + \lambda^2 \cos 2\lambda\alpha = -1. \quad \text{.....(iii)}$$

These equations (ii), (iii) are satisfied at P where

$$x = \pi/\lambda + \beta \quad \text{and} \quad y = -\sin (\pi + \lambda\beta) = \sin \lambda\beta.$$

(iii) becomes

$$-\lambda^2 \sin \lambda\beta \sin \lambda\alpha + \lambda^2 \cos 2\lambda\alpha = -1 = -\lambda^2 \cos \lambda\alpha \cos \lambda\beta, \text{ by (i).}$$

Thus $\cos(\lambda\alpha + \lambda\beta) + \cos 2\lambda\alpha = 0,$

so that $(\lambda\alpha + \lambda\beta) + 2\lambda\alpha = \pi$

or $3\alpha + \beta = \pi/\lambda,$

which is $3AL + BM = AB.$

Substituting in (ii), using $\sin\lambda\beta = \sin 3\lambda\alpha$ and $\lambda^2 = -2/(\cos 4\lambda\alpha + \cos 2\lambda\alpha)$ and putting $\theta = 2\lambda\alpha$, we eventually arrive at the equation

$$\operatorname{cosec}\theta - 2\cot\theta = 1/(\pi - \theta).$$

This is soluble by tables, and such solution does not take long. Since $\lambda\alpha$ is between 30° and 45°, θ is between 60° and 90°. Trying $\theta = 70°$ and 80° we get that the left-hand side minus the right-hand side = ·1836 and −·0897, and so guess 76° or 77°. Trying these, we get differences ·0190 and −·0084, from which we guess 76° 42′. Trying 76° 36′, 76° 42′, 76° 48′, we find that 76° 42′ gives a difference −·0001 and so is correct to the nearest minute. This gives $\lambda\alpha = 38°\ 21'$, $\lambda\beta = 64°\ 57'$, and finally $\lambda^2 = \sec 38°\ 21' \sec 64°\ 57'$, $\lambda \simeq 1{\cdot}74$.

On the whole, the problem just solved cannot be recommended for any pupils except the best of our mathematical specialists, but there are plenty of easy questions available.

For example, Fig. 1 suggests :

(i) What is the angle between the curves where they cross?

(ii) What is the area between the curves?

(iii) Show that the tangents and normals at B and C make a square of area $\pi^2/8$, and find the coordinates of its other vertices.

For Fig. 2 we have :

(i) Prove that the common normal is $x + y = 1$.

(ii) Show that the tangent to $y = e^x$ from the origin touches the curve at $(1, e)$.

(iii) Show that the tangents to $y = \log x$ at the points where $y = \pm 1$ meet where $y = 2/(e^2 - 1)$.

Again, the other trigonometrical curves will provide examples of varying difficulty. If the curves for $y = \sec x$ and $y = \operatorname{cosec} x$ are drawn, it will be seen that the branches above and below the x-axis have common tangents and normals. The branch of $y = \sec x$ between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ has a common tangent and a common normal with the branch between $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$, and symmetry shows that both these pass through $(\frac{1}{2}\pi, 0)$. The subtangent and subnormal being $\cot\theta$ and $\sec^2\phi\tan\phi$ respectively, we get equations

$$\cot\theta - \theta = \tfrac{1}{2}\pi, \quad \sec^2\phi\tan\phi + \phi = \tfrac{1}{2}\pi,$$

to determine the points of contact, which are not far from $\pm 30°$. Similarly the cosecant curve between 0 and π and the secant curve between $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$ have common tangent and normal, both passing through $(\frac{3}{4}\pi, 0)$ and in the equations $\frac{1}{2}\pi$ is replaced by $\frac{1}{4}\pi$. For the tangent curve one may inquire why a normal to $y = \tan x$ from $(\frac{1}{2}\pi, 0)$ is normal to the two branches and find an equation for the angle concerned.

Here is another example. For the curves $y = a^2x^3 - 3bx$

(i) there are no tangent-normals if $b < \frac{4}{9}$, (all a) ;

(ii) there are two pairs of tangent-normals if $b > \frac{4}{9}$, (all a) ;

(iii) if $b = \frac{4}{9}$, there is a *unique pair of tangent-normals*.

For the curve $y = a^2x^3 - \frac{4}{3}x$, the point of contact of the tangent-normal is

given by $ax = \frac{1}{6}\sqrt{10} \simeq \cdot 527$ and its gradient is $-\frac{1}{2}$. So for all these curves, given by different values of a, *the tangent-normals are all parallel.* Their points of contact lie on a straight line through the origin whose equation is $3x + 4y = \frac{1}{2}x\sqrt{10}$, and the points where these lines are normals also lie on a straight line through the origin.

Again, for the curve, or curves, $y^2 = e^{2x}$, if the line $y = x + 1$ cuts the curve where $x = -2x_1$, then the normal to $y = -e^x$ where $x = -x_1$ is tangent to $y = e^x$ where $x = x_1$.

Possibly some teachers will dislike the mixture of accurate work with approximate solution of equations which is typical of the type of problem suggested, but this mixture seems to fit the modern slogan "elementary mathematics is a single subject".

It may be added that anyone who "discovers" any property of tangents or normals to conics is quite certain to find out on enquiry that it has been discovered and listed long ago, but it is quite possible that no one has yet taken the trouble to find out which tangents of $y = \sin 3x$ are also normals.

C. O. T.

CORRESPONDENCE.

THE ORDER OF TEACHING MATHEMATICAL SUBJECTS.

To the Editor of the *Mathematical Gazette.*

SIR,—If a suggested reform of mathematical teaching in schools involved a change of order, it would be impracticable, if only because it would so upset any pupil coming to or going from his school, unless some assignment plan were operating by which the order of work could be varied. This seems to me to be a matter of considerable importance, since some experiments in "coaching" have convinced me that our traditional order could, in several places, be changed with great advantage.

Without perhaps realising it, we tend to follow the order proper to a systematic academic treatise; so a teacher of language might spend the first term studying the noun, then the pronoun, then the adjective, followed by a term or two on the verb and the adverb, the conjunction . . . ending with sentences and word order. This is possibly very good for those who already know a good deal of the language and can speak it reasonably well, but it would not do for beginners. So too the traditional order may be sound for mathematical work at the University level, but not always (I do not say "never") for beginners at school.

I am sure that many members would welcome the opinion of the Teaching Committee on this topic.

Yours, etc., C. DUDLEY LANGFORD.

1647. There is then no sufficient ground for admitting Mr. Locke's celebrated definition of wit, which he makes to consist in the finding out striking and unexpected resemblances in things as so to make pleasant pictures in the fancy, while judgment and reason, according to him, lie in the clean contrary way, in separating and nicely distinguishing those wherein the smallest difference is to be found. On this definition Harris, the author of *Hermes*, has very well observed that the demonstrating the equality of the three angles of a right-angled triangle to two right ones, would, upon the principle here stated, be a piece of wit instead of an act of the judgment or understanding, and *Euclid's Elements* a collection of epigrams.—William Hazlitt, *Lectures on the English Comic Writers*, I.

ON LEMOINE'S ELLIPSE.

By Vincenzo G. Cavallaro.

1. In a triangle ABC of area Δ, $BC=a$, $CA=b$, $AB=c$ are the sides ; O the centre and R the radius of the circumcircle ; ρ_1 and ρ_2 the radii of the first and second Lemoine circles ; σ the radius of the Brocard circle ; O_9 the centre of the nine-points circle ; Ω, Ω' the Brocard points and ω the Brocard angle ; K the Lemoine point ; G the centroid ; H the orthocentre ; m_a, m_b, m_c the medians ; h_a, h_b, h_c the altitudes ; $AL=s_a$, $BS=s_b$, $CT=s_c$ the symmedians corresponding to the sides a, b, c ; 2α and 2β the axes of the Brocard ellipse of foci Ω, Ω' and centre V ; $2x$ and $2y$ the axes of the Lemoine ellipse of foci G, K and centre ϕ ; τ and τ' the Torricelli segments (the invariant segments which join a vertex of ABC to the vertices of the equilateral triangles constructed on the opposite side, internally and externally to the fundamental triangle) ; $p(X)$ the power of a point X with respect to the circle $O(R)$.

The lengths of the symmedians and of the segments which the point K determines on them can be found by the theorem of van Aubel and Stewart ; we have known results

$$AL=s_a=2bcm_a/(b^2+c^2), \quad AK=2bcm_a/(a^2+b^2+c^2),$$
$$KL=2bca^2m_a/(b^2+c^2)(a^2+b^2+c^2) ; \qquad \ldots\ldots(1)$$

with similar formula for the other symmedians BS, CT.

2. We have

$$\rho_2=R\tan\omega=R\,.\,4\Delta/(a^2+b^2+c^2)=abc/(a^2+b^2+c^2) \qquad \ldots\ldots(2)$$

and, by the formulae (1),

$$a\,.\,AK/m_a=b\,.\,BK/m_b=c\,.\,CK/m_c=2\rho_2=2R\tan\omega. \qquad \ldots\ldots(3)$$

If X, Y, Z are the orthogonal projections of K on BC, CA, AB, then by the sine formula, namely, $2R=a/\sin A=\ldots$, we have

$$YZ/a\,.\,AK=ZX/b\,.\,BK=XY/c\,.\,CK=1/2R, \qquad \ldots\ldots(4)$$

whence, by (3),

$$YZ/m_a=ZX/m_b=XY/m_c=\rho_2/R=\tan\omega, \qquad \ldots\ldots(5)$$

which proves the known result that the triangle XYZ of area Θ, the pedal of the Lemoine point, is similar to the triangle of area Φ formed by the medians of the fundamental triangle. (This can be proved geometrically.)

It is known that $\Phi=\frac{3}{4}\Delta$; hence, by (5),

$$\tfrac{3}{4}\Delta=\Phi=\Theta\cot^2\omega\,; \quad \Theta=\tfrac{3}{4}\Delta\tan^2\omega. \qquad \ldots\ldots(6)$$

Thence we have, by (3),

$$AK=m_a\tan\omega\operatorname{cosec}A, \quad BK=m_b\tan\omega\operatorname{cosec}B,$$
$$CK=m_c\tan\omega\operatorname{cosec}C. \qquad \ldots\ldots(7)$$

3. Since $2m_a=3AG$, ... we have, squaring and summing the formulae (7) and using the known result

$$\operatorname{cosec}^2A+\operatorname{cosec}^2B+\operatorname{cosec}^2C=\operatorname{cosec}^2\omega,$$

the following elegant relation

$$\Sigma=\left(\frac{AK}{AG}\right)^2+\left(\frac{BK}{BG}\right)^2+\left(\frac{CK}{CG}\right)^2=\left(\frac{3}{2}\right)^2\sec^2\omega, \qquad \ldots\ldots(A)$$

connecting *the distances of the foci G, K of the Lemoine ellipse from the vertices of the fundamental triangle.*

4. The result (A) can be expressed differently. It is known* that

$$\sec^2\omega = 4R/(3R + 2O_9V) = R^2/(R^2 - \alpha^2) = 2R/(2R - \beta)$$
$$= 4\alpha^2/(4\alpha^2 - \beta^2) = 4{\rho_1}^2/(4{\rho_1}^2 - {\rho_2}^2) = {\rho_2}^2/\alpha^2$$
$$= 4{\rho_1}^2/R^2 = 4(R^2 - \sigma^2)/3R^2.$$

Thus, in particular,

$$\Sigma = \frac{(3\alpha)^2}{(2\alpha)^2 - \beta^2} = 3\left\{1 - \frac{\sigma^2}{R^2}\right\} = 3\,.\,\frac{3{\rho_2}^2}{4\alpha^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(B)$$

that is to say, *the sum Σ is expressed in terms of the Brocard axes, or in terms of the radii σ and R, etc.*

It is also known † that $p(K) = 3{\rho_2}^2$, $p(\Omega) = p(\Omega') = 4\alpha^2$; hence, by virtue of the formula (B),

$$\Sigma = 3p(K)/p(\Omega), \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(C)$$

expressing Σ in terms of the powers of the points K, Ω with respect to the circle $O(R)$.

Since ‡ $$OV/O_9V = R/\sigma = \tau/\tau' + \tau'/\tau,$$

we have, from (B),

$$\Sigma = \left(\frac{AK}{AG}\right)^2 + \left(\frac{BK}{BG}\right)^2 + \left(\frac{CK}{CG}\right)^2 = 3\left(\frac{\tau^4 + \tau^2\tau'^2 + \tau'^4}{\tau^4 + 2\tau^2\tau'^2 + \tau'^4}\right), \ldots\ldots\ldots\ldots(D)$$

and *Σ is expressed in terms of the Torricelli segments.*

5. Generally, if F and F' are the foci of a conic inscribed in a triangle and if 2ξ is the non-focal axis, Laguerre's formula § states that

$$p(F)\,.\,p(F') = 4\xi^2R^2.$$

For the Lemoine ellipse we have

$$p(K) = 3{\rho_2}^2 = 3a^2b^2c^2/(a^2 + b^2 + c^2)^2,\ p(G) = (a^2 + b^2 + c^2)/9$$

and thus, by the formulae (2) of paragraph 2, we deduce that

$$p(K)\,.\,p(G) = \tfrac{1}{3}a^2b^2c^2/(a^2 + b^2 + c^2) = \tfrac{1}{3}abc\rho_2 = \tfrac{4}{3}\Delta R\rho_2$$
$$= \tfrac{4}{3}\Delta R^2\tan\omega = 4y^2R^2,$$

whence $$3y^2 = \Delta\tan\omega, \quad 2y = \frac{2}{\sqrt{3}}\sqrt{(\Delta\tan\omega)},$$

$$\Delta = 3y^2\cot\omega = \tfrac{3}{2}y^2(a/h_a + b/h_b + c/h_c) = 3(y^2/\beta)\sqrt{(4\alpha^2 - \beta^2)},$$
$$= \tfrac{1}{2}\sqrt{3}\,.\,y\sqrt{(a^2 + b^2 + c^2)} = y\sqrt{({m_a}^2 + {m_b}^2 + {m_c}^2)}. \ldots\ldots\ldots\ldots\ldots\ldots(E)$$

We have then, by formulae (6),

$$3y = \sqrt{(3\Delta\tan\omega)} = \sqrt{(4\Phi\tan\omega)} = \sqrt{(4\Theta\cot\omega)} = \sqrt{\{4\sqrt{(\Theta\Phi)}\}}, \ldots\ldots\ldots(F)$$

that is, *the non-focal axis of Lemoine's ellipse is expressed in terms of the areas Θ, Φ* (see paragraph 2).

6. It is known that

$$2\alpha = 2R\sin\omega = abc/\sqrt{(b^2c^2 + c^2a^2 + a^2b^2)},$$
$$2\beta = 4R\sin^2\omega = 4\Delta abc/(b^2c^2 + c^2a^2 + a^2b^2).$$

Thus $$\Delta = \tfrac{1}{2}\beta(bc/a + ca/b + ab/c).$$

* V. G. Cavallaro, *Bull. École Polyth. de Timisoara*, X, 1941, n. 1–2.

† V. G. Cavallaro, *Bull. École Polyth. de Timisoara*, VIII, 1939, n. 3–4.

‡ V. G. Cavallaro, *Mathesis*, 1938, p. 293 (Bruxelles).

§ *Nouvelles Annales*, 1879, p. 24.

Now, by a formula from (E), this gives

$$\frac{\beta}{y} = \frac{\sqrt{\{3(a^2+b^2+c^2)\}}}{(bc/a + ca/b + ab/c)}, \quad \text{.....(G)}$$

giving the ratio of the non-focal axes of the Brocard and Lemoine ellipses.

Since $1/4\alpha^2 = 1/a^2 + 1/b^2 + 1/c^2 = (h_a^2+h_b^2+h_c^2)/4\Delta^2$,

we deduce that $\Delta = \alpha \cdot \sqrt{(h_a^2+h_b^2+h_c^2)}$ .....(H)

and we have, from (E),

$$\Delta = y \cdot \sqrt{(m_a^2+m_b^2+m_c^2)}, \quad \text{.....(I)}$$

so that $\alpha/y = \sqrt{\{(m_a^2+m_b^2+m_c^2)/(h_a^2+h_b^2+h_c^2)\}}$, ...(L)

a formula which gives the ratio of the focal axis of Brocard's ellipse to the non-focal axis of Lemoine's ellipse.

7. If we know the distances of any point P in the plane of ABC from A, B and C, we can calculate the distance of P from the centroid G by means of the familiar formula

$$GP^2 = \tfrac{1}{3}(AP^2+BP^2+CP^2) - \tfrac{1}{9}(a^2+b^2+c^2).$$

Since the formulae (1) gives the distances of K from A, B, C we may calculate the distance GK between the foci G and K of the Lemoine ellipse, and hence the focal axis $2x$ of this ellipse may be found by the fact that $4x^2 = 4y^2 + GK^2$.

8. If J is the centre of the Brocard circle, then considering the points O, G, K, Ω, Ω', J, ϕ, V, O_9, H... we know that J is the midpoint of $OK = 2\sigma$, ϕ is the midpoint of GK, V is on OK and is the midpoint of $\Omega\Omega'$, and O, G, O_9 and H are on Euler's line, and such that $OO_9 = O_9H$, $GH = 2OG$, etc. Using Stewart's theorem, we can calculate the distances between pairs of these points. Thus $J\phi = \frac{1}{2}OG$, and in the triangle $O\phi K$ we can calculate the distance $V\phi$ between the centres of the Brocard and Lemoine ellipses. For these calculations it is enough to remark * that

$$OG^2 = R^2 - \tfrac{1}{9}(a^2+b^2+c^2), \quad OK = 2\sigma,$$

$$OV = R^2\sigma/2\rho_1^2, \quad KV = \sigma\rho_2^2/2\rho_1^2, \quad O_9V = R\sigma^2/2\rho_1^2,$$

and that $O\phi$ can be calculated as the median to the side GK of the triangle OGK whose sides we already know.

V. G. C.

* *See* Cavallaro, previous footnotes.

1648. " Oh, come now, Professor, that won't do! " Hanslet protested. " It's perfectly obvious that in this case at least there are only three alternatives, suicide, accident, or murder. Mawsley's death must have been due to one of them."

" In space it is perfectly obvious that there are only three dimensions," Dr. Priestley replied. " Length, breadth, and height. Yet a mathematician can prove, to his own satisfaction, at least, that a fourth exists. However, we need not labour that point. . . ." . . . Jimmy, considering the matter next day, came to the conclusion that for once the Professor was looking for a mare's nest. It was all very well for him to refer obscurely to the fourth dimension, but this was a matter not of mathematics, but of hard common sense.—John Rhode, *Death in Harley Street*, pp. 99 and 104. [Per Mr. Arthur French.

SOME PROPERTIES OF THE ISOSCELES TETRAHEDRON.

By John Leech.

The object of this paper is to discuss some properties of the Isosceles Tetrahedron, in which each edge is equal to its opposite. The last three theorems are possibly new, the remainder are included for their own interest or for completeness in listing elementary results; in the latter case, proofs are not given.

Notation used: $ABCD$ is the tetrahedron, $AB=CD=l$, $AC=DB=m$, $AD=BC=n$. (l, m and n are used in preference to a, b and c as the latter often lead to confusion when dealing with a tetrahedron $ABCD$ and are often kept to denote the areas of faces, which is more systematic.) R, r and r' are the circumradius, inradius and eradius of the tetrahedron, R_0, r_0 and r_1, r_2, r_3 are the circumradius, inradius and eradii of a face. G is the centroid of the tetrahedron, and the "centres" of the faces are distinguished by suffixes corresponding to the opposite vertices. Δ is the area of a face. Properties of the triangle are assumed throughout.

Theorem 1. The faces are congruent acute-angled triangles. The vertices are congruent trihedra, the sum of the face angles at each being two right angles.

Theorem 2. The tetrahedron is self-congruent in the following ways:

$$ABCD\equiv BADC\equiv CDAB\equiv DCBA.$$

This has many consequences by symmetry.

Theorem 3. The joins of the mid-points of pairs of skew edges are perpendicular to those edges and to each other. The squares of their lengths are:

$$\tfrac{1}{2}(m^2+n^2-l^2),\ \tfrac{1}{2}(n^2+l^2-m^2),\ \tfrac{1}{2}(l^2+m^2-n^2).$$

Theorem 4. The volume of the tetrahedron is given by:

$$72V^2=(m^2+n^2-l^2)(n^2+l^2-m^2)(l^2+m^2-n^2).$$

Theorem 5. The circumcentre and incentre of the tetrahedron coincide with the centroid. The centres of espheres "escribed to faces" are the reflections of the vertices in the centroid. The espheres "escribed to edges" are of infinite radius.

$$R^2=\tfrac{1}{8}(l^2+m^2+n^2),\ r=3V/4\Delta,\ r'=3V/2\Delta.$$

The altitude of the tetrahedron is $3V/\Delta$.

Theorem 6. $$16\Delta^2R^2=l^2m^2n^2+9V^2.$$

This can be proved either algebraically, expressing each side as a symmetric polynomial in l, m, n, or, more directly, as follows. The insphere touches each face at its circumcentre, so that

$$\begin{aligned}R^2&=R_0^2+r^2\\&=(lmn/4\Delta)^2+(3V/4\Delta)^2,\end{aligned}$$

and $$16\Delta^2R^2=l^2m^2n^2+9V^2.$$

This may be written: $2\Delta^2(l^2+m^2+n^2)=l^2m^2n^2+9V^2$.

Theorem 7. If ρ_0 is the radius of the self-polar circle of a face, then

$$\rho_0^2+4r^2=0.$$

For
$$\begin{aligned}R_0^2+2\rho_0^2=O_aH_a^2&=9O_aG_a^2\\&=9(GG_a^2-GO_a^2)\\&=9\{\tfrac{1}{9}R^2-(R^2-R_0^2)\}\\&=9R_0^2-8R^2.\end{aligned}$$

Hence $$2\rho_0^2 = 8R_0^2 - 8R^2$$
$$= -8r^2$$

and $$\rho_0^2 + 4r^2 = 0.$$

This confirms that the faces must be acute-angled triangles, as only then is their self-polar radius imaginary so that the inradius of the tetrahedron can be real.

Theorem 8. There is a sphere which touches the three edges of a face internally and the remaining three externally.

It is easily seen, for instance, that the incircle of BCD touches CD, DB, BC in the same points as those in which they are touched by the ecircles of ACD, ADB, ABC remote from A, and that pairs of these ecircles touch AB, AC, AD in the same points. Thus these four circles touch each other and they therefore lie on a sphere which touches the six edges as required. There is one such sphere opposite each vertex; clearly they are of equal radius.

There is a sphere touching all six edges of a tetrahedron internally only if the sums of the lengths of pairs of skew edges are equal. If such a tetrahedron is isosceles, it must be regular. Discussion of this sphere is outside the scope of this paper.

Theorem 9. The four incentres and twelve ecentres of the faces are vertices of four tetrahedra congruent to the given tetrahedron.

Let the centres of the incircle and ecircles mentioned in the proof of the previous theorem be I_a, I_1, I_2, I_3. The normals to the faces at these points are concurrent in the centre of the sphere touching the six edges and containing these circles. Let this centre be J and let the foot of the perpendicular from J on to CD be P.

Lemma. If ϕ is the dihedral angle of the tetrahedron on CD, then

$$\cos\phi = l^2(m^2 + n^2 - l^2)/4\Delta^2 - 1.$$

Let L, L' be the mid-points of AB, CD, and let Q be the foot of the perpendicular from L' on to a line through B parallel to CD. LL' bisects the dihedral angle on CD, hence

$$\angle QL'L = \tfrac{1}{2}\phi,$$

and $$\cos\tfrac{1}{2}\phi = LL'/QL'$$
$$= LL'/(2\Delta/l)$$
$$= l \,.\, LL'/2\Delta.$$

Thus $$\cos\phi = 2\cos^2\tfrac{1}{2}\phi - 1$$
$$= 2(l \,.\, LL'/2\Delta)^2 - 1$$
$$= l^2(m^2 + n^2 - l^2)/4\Delta^2 - 1,$$

which proves the lemma.

Applying the cosine formula, we have

$$\begin{aligned} I_aI_1^2 &= I_aP^2 + I_1P^2 + 2I_aP \,.\, I_1P\cos\phi \\ &= r_0^2 + r_1^2 + 2r_0r_1\cos\phi \\ &= (r_1 - r_0)^2 + 2r_0r_1l^2(m^2 + n^2 - l^2)/4\Delta^2 \\ &= 4\Delta^2\left(\frac{1}{m+n-l} - \frac{1}{l+m+n}\right)^2 + \frac{8\Delta^2 \,.\, l^2(m^2+n^2-l^2)}{(m+n-l)(l+m+n)\,.\,4\Delta^2} \\ &= 4\Delta^2\left(\frac{2l}{(m+n-l)(l+m+n)}\right)^2 + \frac{2l^2(m^2+n^2-l^2)}{(m+n-l)(l+m+n)} \end{aligned}$$

$$= \frac{l^2(l+m+n)(m+n-l)(n+l-m)(l+m-n)}{(m+n-l)^2(l+m+n)^2} + \frac{2l^2(m^2+n^2-l^2)}{(m+n-l)(l+m+n)}$$

$$= \frac{l^2(n+l-m)(l+m-n)+2l^2(m^2+n^2-l^2)}{(m+n-l)(l+m+n)}$$

$$= l^2\,\frac{(m+n)^2-l^2}{(m+n)^2-l^2}$$

$$= l^2,$$

and $$I_aI_1 = l.$$

Similarly $$I_2I_3{}^2 = r_2{}^2 + r_3{}^2 - 2r_2r_3\cos\phi$$
$$= l^2$$

by a similar calculation, and

$$I_2I_3 = l.$$

Similarly the remaining edges of $I_aI_1I_2I_3$ are equal to the corresponding edges of $ABCD$, and the theorem is proved.

Theorem 10. The spheres which touch the edges of the tetrahedron are of radius

$$\rho = 2\Delta^2/3V.$$

With the notation of theorem 9, I_aPI_1J is cyclic and $\angle I_aPI_1 = \pi - \phi$.

Hence $$I_aI_1 = \rho\sin\phi$$
$$= \rho h/h_a,$$

where h, h_a are the altitudes from A of the tetrahedron and of ACD, hence

$$\rho = lh_a/h$$
$$= l\,.\,\frac{2\Delta}{l}\Big/\frac{3V}{\Delta}$$
$$= 2\Delta^2/3V.$$

Corollary. $$\Delta = 2r\rho = r'\rho = \tfrac{1}{2}h\rho.$$

Theorem 11. The spheres which touch the edges of the tetrahedron are orthogonal to the circumsphere.

We saw in Theorem 8 that these spheres contain the incircles and ecircles of the faces. Thus they touch the edges at points whose distances from the vertices on those edges are $\frac{1}{2}(l+m+n)$, $\frac{1}{2}(m+n-l)$, $\frac{1}{2}(n+l-m)$, $\frac{1}{2}(l+m-n)$.

Thus $$AJ^2 = \rho^2 + \tfrac{1}{4}(l+m+n)^2, \text{ etc.,}$$

and $$AJ^2 + BJ^2 + CJ^2 + DJ^2 = \rho^2 + \tfrac{1}{4}(l+m+n)^2 + \rho^2 + \tfrac{1}{4}(m+n-l)^2$$
$$+ \rho^2 + \tfrac{1}{4}(n+l-m)^2 + \rho^2 + \tfrac{1}{4}(l+m-n)^2$$
$$= 4\rho^2 + l^2 + m^2 + n^2$$
$$= 4\rho^2 + 8R^2.$$

Now $$AJ^2 + BJ^2 + CJ^2 + DJ^2 = AG^2 + BG^2 + CG^2 + DG^2 + 4JG^2$$
$$= 4R^2 + 4JG^2.$$

Combining, we obtain $$JG^2 = R^2 + \rho^2.$$

Thus the square on the join of the centre of the circumsphere to the centre of a sphere touching the edges is equal to the sum of the squares on their radii, and the spheres are orthogonal. J. L.

APPLICABILITY AND THE TEACHING OF GEOMETRY.

By D. E. Littlewood.

The mistakes of great men leave long shadows in history. Euclid's proofs in the main are elegant and concise, and apart from style of presentation, with slight alterations form the basis for most modern geometrical texts. Exception must be taken, however, to the earlier introductory propositions. These have been found unsuitable for teaching to children. Some have sought to explain this away by saying that Euclid wrote for mature adults, but that children of 11 or 12 cannot comprehend logical proofs. From other eminent authorities, however, come criticisms of the proofs themselves. Thus Russell says that Euclid's proposition I, 4, is a tissue of nonsense. Undoubtedly Euclid's earlier proofs fall far short of perfection, but, perversely enough, the criticisms which have been levelled at them are even more mistaken than are Euclid's proofs.

Since the time of Euclid much has been learnt concerning the nature of geometry. Quite early it was observed that there were other varieties of geometry besides that of Euclid. The key to the whole riddle was provided by Klein in the *Erlanger Program*. He showed that the distinguishing feature of any geometry is its *group of correspondences*.

Strange that this great truth has not hitherto been applied to the foundations of Euclid, for it shows up truth from error with astonishing clarity. If Russell had been really familiar with Klein's *Erlanger Program* he could hardly have made the grievous error inherent in his criticism of Euclid I, 4.

But is this of concern to a practising schoolmaster? It is, indeed, for the method of teaching geometry depends on the outcome. It is often believed that logical soundness and rigour are hard and difficult achievements that only mature and intelligent minds can win. The truth is opposed to this. Like a nut being turned onto a threaded rod, the right way is easy and simple, it is the wrong way which is difficult and hard, like driving the nut cross-threaded on the rod.

What is right and correct in the foundations of geometry turns out to be in complete accord with the intuition and previous practical experience of a child, so that 11- and 12-year-olds find confirmation and coordination of their inherent knowledge rather than a difficult new subject.

The crux of the matter turns on the proof of Euclid I, 4. The idea behind the proof is very simple, it is the moving of one triangle so that it fits onto the other, point on point and line on line. But is this method sound? Criticisms are quoted from the 1923 *Report* of the Mathematical Association. Russell wrote bluntly: "The fourth proposition is a tissue of nonsense." Heath gives a summary of other criticisms to similar effect.

"Since geometry is concerned with empty space it would at least be strange if it was necessary to have recourse to the real motion of bodies for a definition of equality and for the proof of properties of immovable spaces. If we say that two bodies are equal when they can be superimposed by means of a *movement without deformation*, we are committing a *petitio principii*; the definition of rigidity assumes a criterion for the equality of spaces, and if the only criterion available is derived from superposition, then there is no escape from the vicious circle."

The essential point that all the philosophers appear to have missed is that the denial of the principle of superposition does not merely invalidate one theorem, but destroys the whole structure of Euclidean Geometry.

To prove that the line AB is equal to the line CD any child will move a rigid body, say his ruler, from coincidence with the one into coincidence with

the other. If the philosophers describe this test as nonsense, they must needs, to justify themselves, supply an alternative test. It requires very little time and thought to reach the inevitable conclusion that no such independent test exists. Einstein, in both his Special and General Theories, found it necessary to make use of measuring rods and clocks in order to explore space time. The complete failure to compare the lengths *AB* and *CD* without recourse to rigid bodies provides in itself a *reductio ad absurdam* for the attitude of the philosophers. But clearly some constructive criticism is here required to restore the structure.

Readers of Levi Civita's *Absolute Differential Calculus* will recognise from his development that points in abstract space do not in themselves possess any metrical properties at all, *except they be first defined*. The philosophers assumed the metrical properties to be inherent in the space in their own right, and this is the source of the dilemma. But to determine which is the right and appropriate method of defining the metric, it is necessary to appeal to Klein's concept of a geometry as an aggregate of points with a group of correspondences.

There are various different systems of geometry, projective, affine, Euclidean, elliptic, hyperbolic, and for every geometry there is a set of one-one correspondences which necessarily display the group property. The concept is perhaps most familiar in projective geometry, where the correspondence is obtained in comparing two projective figures obtained the one from the other by a sequence of perspectives. Expressed algebraically the group of such correspondences, or the projective group, is the group of all non-singular linear transformations in the appropriate number of variables, one more than the number of dimensions. This is the simplest possible group of correspondences for any geometry.

Taking the subgroup which leaves invariant a fixed line called the *line at infinity*, affine geometry is obtained. A further restriction so as to leave invariant a quadratic function, the metric, of the coordinate differences of any two arbitrary points, or equivalently so as to leave invariant a pair of points called the *circular points* on the line at infinity, leads to the Euclidean group and Euclidean Geometry. It is a complicated group, the more complicated because, in common with affine geometry it includes *step transformations*, and thus leads to representations which are semi-reducible without being completely reducible.

If the Euclidean group is so much more complicated than the projective group, why is it that Euclidean geometry is thought suitable for teaching to young children while projective geometry is not? The answer is incontrovertible and most enlightening. *The Euclidean group happens to be identical with the group of motions of a rigid body*! Children have already had practical experience of the motions of a rigid body, and have obtained an intuitive understanding which is based on this practical experience!

This is the principle of applicability. To fail to use it is to fling away the whole body of experience which the child brings spontaneously to the classroom.

Russell came very close to the truth on another occasion when, still criticising superposition (see 1923 *Report*, p. 33), he writes: "Motion is a certain class of one-one relations." What a pity that he did not recognise this class of relations as the Klein group which defines the geometry!

It must be considered, however, how the metric can be defined without the vicious circle, the *petitio principii* that Heath describes. Given an affine geometry, there is a certain arbitrariness in the manner in which a metric is defined so as to obtain from it a Euclidean geometry. It follows from this arbitrariness that it is not possible to define a single rigid body, for its

behaviour must depend on the particular metric chosen. It is, however, possible to define a *system of rigid bodies*. A system of movable bodies is a system of rigid bodies if, when equality is defined by the possibility of superposition, no length or area can ever be equal to a part of itself. It is just this consistency of measurement which defines rigidity. It is theoretically possible to have two quite different systems of bodies, each system rigid relative to itself, but such that bodies of the one system are deformed in their motion when compared with the other system. Each system of rigid bodies would then define its own system of Euclidean geometry.

It is not necessary for either the schoolmaster or the child to consider these niceties, however, which are here expressed only to meet the objections of hypercritics.

If the fundamental property of a geometry is its group of correspondences, then the fundamental concept of Euclidean geometry is the comparison of geometrical figures by the motions of rigid bodies from one position to another. The child comprehends this intuitively, and without prompting will compare the lengths of the two lines with a ruler, just a suitable rigid body. Every geometrical instrument, dividers, compasses, set square, protractor, is a rigid body which is moved to different positions in the appropriate way.

There is, however, a fault in Euclid's proof of the proposition I, 4. There are two methods of comparing two lengths, say the lengths of two pencils. Each can be measured against, say, a ruler, or alternatively a direct comparison can be made by placing the two pencils together. The second method is not always available, as two parts of the same figure or of the same rigid body cannot be so compared. Unfortunately, Euclid uses the second method rather than the first. This is a distinct flaw in the proof, and is not only a logical mistake but is intuitively confusing. It is not necessary to move one triangle into coincidence with the other, it is only necessary to move a rigid body (say a cut-out piece of cardboard to fix the attention) from coincidence with the one triangle into coincidence with the other. This would avoid part of the criticism which is levelled at the proof. It is an important point in comparing parts of the same figure. Thus, to show that the diagonal of a parallelogram divides the figure into two congruent triangles, it is impossible to move one half into coincidence with the other, for the other half would move away at the same time, like a kitten chasing its tail. But there is no difficulty in moving a rigid body from coincidence with the one half into coincidence with the other.

Once a child has been made conscious of this principle of applicability, which implies only a bringing to the conscious level of a subconscious awareness that is already present, then the dullest child can hardly fail to realise that it is sufficient to get two sides and the included angle fitting correctly, or one side and the two angles at its extremities, and then the whole triangle will fit. He will then have grasped thoroughly and completely two fundamental propositions and the understanding of geometry is well begun.

The modification of Euclid's proof mentioned above makes the conclusion more powerful, for it is quite justifiable to refer to a triangle which is *congruent to itself*. Thus for an isosceles triangle ABC with $AB = BC$ it is correct to say that the triangle ABC is congruent to the triangle CBA, two sides and included angle. Hence $\angle C = \angle A$ and the base angles are equal. This simply means that a rigid body fitting ABC can be turned over so as to fit CBA in accordance with the method of proof of Euclid I, 4. The converse follows similarly from the other fundamental theorem.

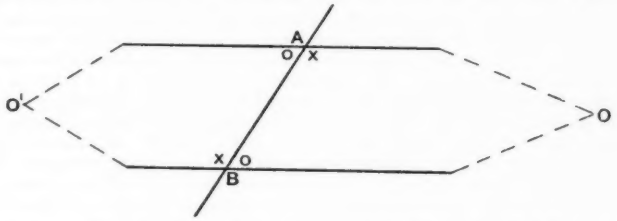
Euclid's proof can now be used for the case of congruence given three sides : it is but one step removed from the principle of applicability.

There remains the presentation of the theorems on parallels. Euclid's treatment here is tortuous, involving the proof of theorems on inequalities which subsequently become useless, and which are of doubtful logical soundness.

It can be made clear that the case of congruence given two angles and the side opposite one cannot be demonstrated without some new assumption, for the method of applicability can be used also for triangles on the surface of a sphere. The triangles formed by two lines of longitude and the equator present an example when two angles and a side opposite one do not result in congruence.

The necessary assumption is best made in the form of Playfair's axiom. Parallel lines are defined as pairs of lines which do not meet. The assumption is made that there is one and only one line which passes through a given point and is parallel to a given line.

Now let a transversal AB cut two lines so that the alternate angles are equal. If the lines are not parallel they will meet at O. Apply the triangle ABO (or a rigid body fitting the triangle) so that A falls on B, B on A and the



triangle is on the other side of the transversal. Then O will fall on another point O'. Thus the lines will meet in two points. Since this is not possible the lines must be parallel.

The remaining proofs of Euclid follow without real difficulty, and the whole foundation is secure.

In conclusion, it must be re-emphasised that the approach here recommended builds directly on the intuition and past experience of the child. This should be sufficient justification without any questions of logic arising. But, in fact, not only is it perfectly rigorous, it is the only approach which is both logically and intuitively sound. D. E. L.

1649. Nothing Like Enough.

Someone has nodded in composing the latest Government advertisement in the " Challenge to British Grit " series.

Headed : " 8d. Isn't Enough," the advertisement reads :

" Eightpence won't buy a shillingsworth of goods. But for every shillingsworth of cotton or rubber, tea or food that we get from abroad, we had only eightpennyworth of exports to offer last year. The rest we are getting on tick.

" Now you see why we must export one-third more this year, and still more after that. We must pay our way—or get less and fare worse."

I offer no prizes for spotting the flaw in this challenge to British arithmetic. —" Peterborough " in the *Daily Telegraph*. [Per Mr. A. W. Ready.]

CENTRAL PROJECTION AND STEREOSCOPIC DIAGRAMS ON THE DIFFERENTIAL ANALYSER.

By A. R. Curtis, J. G. L. Michel and Elizabeth D. Sully.

1. *Introduction.*

A description of the construction and use of the differential analyser has already been given in the *Mathematical Gazette* by Professor D. R. Hartree (Vol. XXII (1938), page 342); this note arose from the consideration of means of depicting results to problems depending on three dimensions. In using the machine on what are essentially three-dimensional problems (*e.g.* orbits and trajectories not lying in a plane), results can usually be presented only in tabular form. As a matter of some interest in this field, consideration was given to the possibility of using the machine itself to draw a perspective diagram of such twisted curves in three dimensions. If this were found practicable it was realised that with two such diagrams, representing the space-figure as seen from two points separated by the distance between the eyes, a stereoscope could be used to reproduce a three-dimensional effect.

It was found possible to draw automatically on the differential analyser the two stereoscopic projections for a curve in three dimensions; six integrators are required to perform the algebra of the homographic transformations involved. Eight integrators were available on the differential analyser used for this investigation; the remaining two were used to provide automatically the coordinates of a point traversing a cylindrical helix which was used as an example of a twisted curve. An outline of the principles employed in using the differential analyser is given in § 3 below. In considering the geometrical aspects of the problem no accessible practical treatment of the general problem of projecting a point from a given vertex on to a given plane was found. A simple vector derivation of the general formula was developed; this is reproduced in § 2.

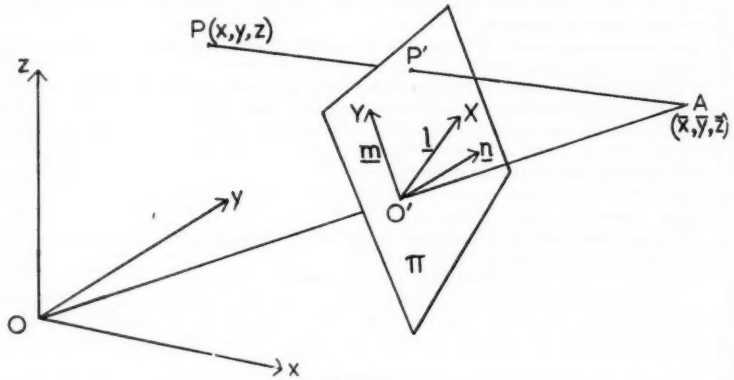


Fig. 1.

2. *Algebraical Formula (homographic transformation) for central projection.*

We refer all points in three-dimensional space to a right-handed rectangular cartesian coordinate system Ox, Oy, Oz. If we are given a vertex of projection $A(\bar{x}, \bar{y}, \bar{z})$ and a plane of projection π on which points are referred to a two-dimensional cartesian system $O'X$, $O'Y$, the problem is to find the X, Y

coordinates of the projection P' in the plane of a general point $P\ (x, y, z)$ in space. In the following proof, where Clarendon type is not used for vectors the notation AP will be assumed to represent the vector of magnitude $|\ AP\ |$ in the sense and direction from A to P.

Let $\mathbf{l}$, $\mathbf{m}$, be unit vectors in the direction of the X-, Y-axes, and let $\mathbf{n}$ be the unit vector normal to the plane so that $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ form a right-handed system. Let O' be the projection of O in the plane π; as a matter of convenience we choose O' as the origin of the X, Y reference axes for the plane (any other choice of origin would involve only a translation).

Let $$OA = \mathbf{d}, \quad O'A = k\mathbf{d} \ (k \text{ a scalar constant}). \qquad \text{.....(1)}$$

Let P', the projection of $P(x, y, z)$ in the plane π from the vertex $A(\bar{x}, \bar{y}, \bar{z})$, have the coordinates X, Y referred to the $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ system (*i.e.* $O'P' = X\mathbf{l} + Y\mathbf{m}$).

Let $$AP' = h\ .\ AP \ (h \text{ a scalar}). \qquad \text{.....(2)}$$

Now $$O'A + AP' = k\mathbf{d} + h\ .\ AP = X\mathbf{l} + Y\mathbf{m}. \qquad \text{.....(3)}$$

Forming the vector product with AP,

$$X(\mathbf{l} \wedge AP) + Y(\mathbf{m} \wedge AP) = k(\mathbf{d} \wedge AP). \qquad \text{.....(4)}$$

The scalar product of (4) with $\mathbf{m}$ and $\mathbf{l}$ respectively gives

$$X(\mathbf{m}, \mathbf{l}, AP) = k(\mathbf{m}, \mathbf{d}, AP), \qquad \text{.....(5a)}$$

$$Y(\mathbf{l}, \mathbf{m}, AP) = k(\mathbf{l}, \mathbf{d}, AP), \qquad \text{.....(5b)}$$

where the notation $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ represents the triple scalar product $\mathbf{a}\ .\ \mathbf{b} \wedge \mathbf{c}$.

Since $\mathbf{l} \wedge \mathbf{m} = \mathbf{n} = -\mathbf{m} \wedge \mathbf{l}$, we deduce from (5a), (5b), after some rearrangement:

$$X = \ k\frac{(\mathbf{m}, AP, \mathbf{d})}{\mathbf{n}\ .\ AP}, \qquad \text{.....(6a)}$$

$$Y = -k\frac{(\mathbf{l}, AP, \mathbf{d})}{\mathbf{n}\ .\ AP}. \qquad \text{.....(6b)}$$

For calculation it is most convenient to express the vector AP as

$$OP - OA = OP - \mathbf{d};$$

then $AP \wedge \mathbf{d} = OP \wedge \mathbf{d}$ and (6a), (6b) become

$$X = \ k\frac{(\mathbf{m}, OP, \mathbf{d})}{\mathbf{n}\ .\ (OP - \mathbf{d})}, \qquad \text{.....(7a)}$$

$$Y = -k\frac{(\mathbf{l}, OP, \mathbf{d})}{\mathbf{n}\ .\ (OP - \mathbf{d})}. \qquad \text{.....(7b)}$$

Let the direction cosines of $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ be λ_1, μ_1, ν_1; λ_2, μ_2, ν_2; λ_3, μ_3, ν_3 respectively (these are related by the usual expressions for a right-handed rectangular coordinate system); since the cartesian components of OP are x, y, z and of $\mathbf{d}$ are $\bar{x}, \bar{y}, \bar{z}$ the explicit cartesian expressions for (7a), (7b) are

$$X = \frac{k\begin{vmatrix} \lambda_2 & \mu_2 & \nu_2 \\ x & y & z \\ \bar{x} & \bar{y} & \bar{z} \end{vmatrix}}{\lambda_3(x - \bar{x}) + \mu_3(y - \bar{y}) + \nu_3(z - \bar{z})}, \qquad \text{.....(8a)}$$

$$Y = \frac{-k\begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 \\ x & y & z \\ \bar{x} & \bar{y} & \bar{z} \end{vmatrix}}{\lambda_3(x - \bar{x}) + \mu_3(y - \bar{y}) + \nu_3(z - \bar{z})}. \qquad \text{.....(8b)}$$

3. *Use of Differential Analyser for Drawing Perspective Diagrams.*

Since it is possible to add and subtract quantities, to multiply by a constant and to obtain the quotient of two varying quantities (Crank (1), Amble (2), Michel (3)), and these are the only arithmetical operations involved in the homographic transformations (8a), (8b), the Differential Analyser can in principle be used to do the arithmetic of the transformations for both diagrams of a stereoscopic pair simultaneously with the solution of the problem. It transpires, however, that at least six integrators are required for doing the two projections simultaneously. Since the Differential Analyser used for this trial contained eight integrators only, two were still available. These were used to manufacture automatically the cylindrical helix

$$\left.\begin{aligned} x &= r\cos t \\ y &= r\sin t \\ z &= pt \end{aligned}\right\}.$$

With t as independent variable, $\sin t$ and $\cos t$ were manufactured automatically by the well-known technique (Hartree, *loc. cit.*, Crank (1)) of using the Differential Analyser to solve the equation of simple harmonic motion,

$$\frac{d^2\phi}{dt^2} + \phi = 0.$$

The technique for obtaining the sums of the quantities appearing in the numerators and denominators of (8a), (8b) is fundamental, and is described in the references already cited. If we choose our two vertices of projection (the positions of the viewer's eyes) to lie on a line parallel to the X-axis, both projections being made on the same plane, it is apparent intuitively or by analysis that the Y-coordinates of a point for both the left-eye and right-eye projections are the same ; only the X-coordinates differ. Thus we have three coordinates to calculate, each calculation being of the form

$$\frac{ax+by+cz}{dx+fy+gz+h}. \qquad \text{.....(9)}$$

Amble, Michel and Crank have described a Differential Analyser technique for obtaining the quotient of two varying quantities on two integrators. This technique was applied to each of the three quotients of form (9), utilising six integrators in all.

A stereoscopic pair of projections of the helix is shown in Fig. 2.

For the original preparation of these diagrams the following data was used:

Equations of helix :

$$\left.\begin{aligned} x &= 3\cos t \\ y &= 3\sin t \\ z &= 0{\cdot}4t - 10 \end{aligned}\right\},$$

i.e. a helix about the z-axis on a right circular cylinder of radius $3''$, base $10''$ below the origin, pitch $0{\cdot}8\pi$.

Data for projection in inch units :

Left vertex :		Right vertex :	
$\bar{x} = 45,$		$\bar{x} = 45,$	
$\bar{y} = -3\frac{3}{4},$		$\bar{y} = 3\frac{3}{4},$	
$\bar{z} = 0.$		$\bar{z} = 0.$	

Direction cosines :

	1	2	3
λ	0	$-\frac{3}{5}$	$\frac{4}{5}$
μ	1	0	0
ν	0	$\frac{4}{5}$	$\frac{3}{5}$

$k = 1$.

These two cases correspond to projections on to a plane having the equation

$$\tfrac{4}{5}y + \tfrac{3}{5}z = 0$$

from two vertices $7\frac{1}{2}''$ apart on a line parallel to the y-axis $36''$ from the plane. The original diagrams were prepared with a view to reduction by $\frac{1}{3}$ during reproduction, the separation of the vertices for the reduced diagrams is thus $2\frac{1}{2}''$, the normal human interpupillary distance, and diagrams are intended for viewing at a distance of $12''$.

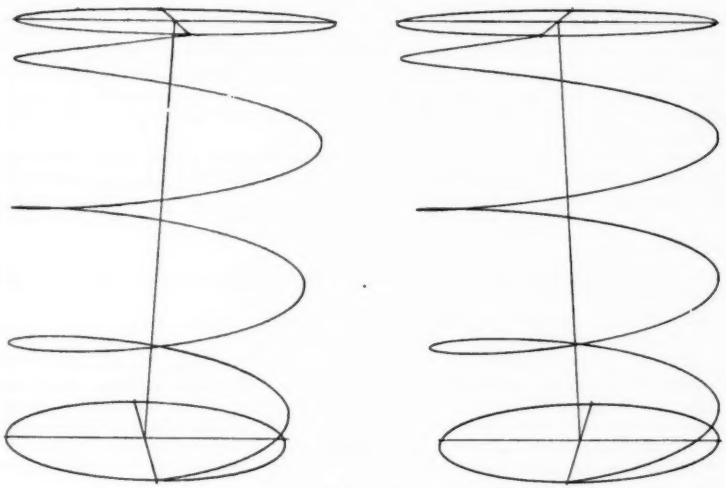


FIG. 2.

Further details on the practical aspects of stereoscopy will be found in works devoted to that subject, such as *Stereoscopic Photography* by A. W. Judge (Chapman and Hall, 1926).

4. In viewing any object having considerable depth (in comparison with the distance from the viewer) it is necessary to use a central projection, to give effect to perspective. Where, however, perspective effects are negligible, two orthogonal projections of the solid figure, projected at angles differing by approximately the angle subtended by the two eyes at the object, will give an adequate effect of solidness when viewed in a stereoscope. The analytic formula for orthogonal projection may be obtained quite simply from first principles, or it can be obtained from formulae (8a), (8b). In Fig. 1 let the plane π pass through the origin O ($k = 1$). Write $\bar{x} = Lr$, $\bar{y} = Mr$, $\bar{z} = Nr$ (L,

M, N direction cosines of OA). Project the point A to infinity by letting $r \to \infty$. Formulae (8a), (8b) become

$$X = \frac{-\begin{vmatrix} \lambda_2 & \mu_2 & \nu_2 \\ x & y & z \\ L & M & N \end{vmatrix}}{\lambda_3 L + \mu_3 M + \nu_3 N} \qquad \text{.....(10a)}$$

$$Y = \frac{+\begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 \\ x & y & z \\ L & M & N \end{vmatrix}}{\lambda_3 L + \mu_3 M + \nu_3 N} \qquad \text{.....(10b)}$$

These are of the form $ax + by + cz$,

which can immediately be formed upon the differential analyser without using integrators.

5. *Summary.*

In suitable cases (10a), (10b) can be used for preparing a stereoscopic pair of diagrams for any three-dimensional problem dealt with on the differential analyser. Where considerations of perspective render this orthogonal projection unrealistic, the central projection, using (8a), (8b), enables satisfactory stereoscopic diagrams to be prepared; if the central projection is used, integrators or some other device for dividing must be used to form the quotients involved.

The work described above has been carried out as part of the research programme of the National Physical Laboratory, and this paper is published by permission of the Director of the Laboratory.

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1650. An 80-year-old Irishman weighing 16 stone was making his first flight from London Airport to Amsterdam. The other passengers were astonished to see him crouched in a most uncomfortable position in the seat of the air liner for the whole of the voyage.

On arrival at Amsterdam the K.L.M. Dutch Airlines captain questioned his elderly passenger on what he thought of flying.

The old man apologised for his weight, and added apologetically: "But I didn't quite sit down during the trip, because I knew my weight might upset the balance of the plane."—*Daily Telegraph*, 19th January, 1950. [Per Mr. P. Vermes.]

1651. According to the obituaries, the late Gordon Selfridge, who concealed the date of his birth with unusual success, was anything from seventy-nine to ninety; but one newspaper managed to credit him with both those ages. Its statement runs: "Retiring at forty in America with £300,000 in 1908, he came to London for adventure. . . . He never revealed his age . . . but it is believed that he was ninety." It is plain, however, that if he was forty in 1908 (even if it was in America!) he must have been seventy-nine when he died the other day. The ability to add two to two and make four out of them is not so general as most people believe.—*Manchester Guardian*, May 12, 1947. [Per Mr. A. Jackson.]

THE GENERAL ISOGONAL CIRCLE.

By Neil Y. Wilson.

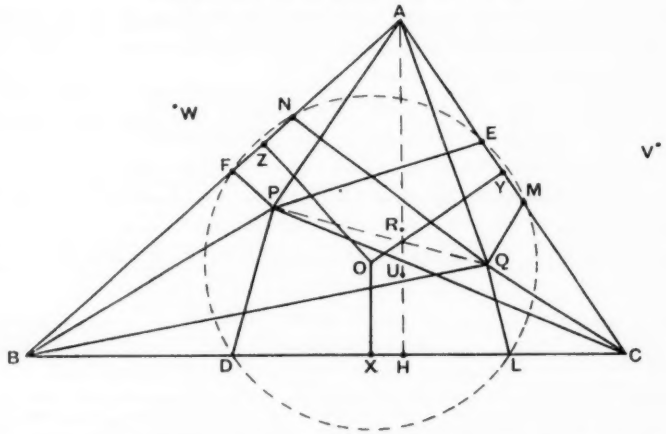
1. *Introduction.* Every geometrical truth has an individual beauty which delights the mind: but the most fascinating theorems, as an eminent mathematician rightly asserts, are those which have simplicity, depth, and obvious generality. I propose to show that nearly all the important circles of the geometry of the triangle—the in-circle, the nine-points circle, the circum-circle, Taylor's circle, Lemoine's circle, and many other useful but nameless circles—are each and all particular cases of one fundamental general circle.

Let P and Q be any two points in the triangle ABC such that

$$\angle PBC = \angle ABQ = B_1 \quad \text{and} \quad \angle PCA = \angle BCQ = C_1.$$

Then $\angle PAB$ and $\angle CAQ$ will be equal. I shall assume the truth of this basic theorem of isogonal theory. It is readily proved by showing that the product of the perpendiculars from P and Q on each side of the triangle ABC is constant. Hence P and Q are isogonal conjugates for the triangle ABC, that is,

$$\angle PAB = A_1, \quad \angle PBC = B_1, \quad \angle PCA = C_1,$$
$$\angle CAQ = A_1, \quad \angle BCQ = C_1, \quad \angle ABQ = B_1.$$



2. *The general isogonal circle.* Let PD, PE, PF make equal angles with the sides,

$$\angle PDC = \angle PEA = \angle PFB = \theta \text{ (say)}.$$

Let QL, QM, QN make the same angle with the sides but in the opposite sense,

$$\angle QLC = \angle QMA = \angle QNB = 180^\circ - \theta.$$

Then D, E, F and L, M, N are six concyclic points.

Proof. $EAFP$ is cyclic and so $\angle FEA = \angle FPA = \theta - A_1$; $MANQ$ is cyclic, and so $\angle ANM = \angle AQM = \theta - A_1$. Thus the triangles FAE, MAN are similar, and hence

$MENF$ is cyclic; .....(i)

similarly $NFDL$ is cyclic, ...(ii)

and $DLME$ is cyclic. .....(iii)

Let X, Y, Z be the midpoints of DL, ME, NF.

Then $$MA \cdot EA = AN \cdot AF,$$

so that $$(b - CM)(b - CE) = (c - NB)(c - FB),$$

and $$b^2 - 2b \cdot CY + CM \cdot CE = c^2 - 2c \cdot ZB + NB \cdot FB$$

or $$b^2 - 2b \cdot CY + DC \cdot LC = c^2 - 2c \cdot ZB + BD \cdot BL.$$

But $$DC \cdot LC - BD \cdot BL = (a - BD)(a - BL) - BD \cdot BL = a^2 - 2a \cdot BX.$$

Thus $$a(2BX - a) + b(2CY - b) + c(c - 2ZB) = 0;$$

hence $$(BX + XC)(BX - XC) + (CY + YA)(CY - YA) + (AZ + ZB)(AZ - ZB) = 0,$$

and so $$BX^2 - XC^2 + CY^2 - YA^2 + AZ^2 - ZB^2 = 0.$$

Hence the perpendiculars to the respective sides through X, Y, Z are concurrent at O, say.

Thence, by (i), $$OM = OE = ON = OF,$$

and, by (ii), $$ON = OF = OD = OL.$$

Therefore O is the centre of a circle which passes through the six points, D, E, F, and L, M, N.

3. *The radius of the isogonal circle.*

$$2DX = BL - BD.$$

Thus $$2DX \sin\theta = QB \sin(\theta - B_1 + B) - PB \sin(\theta - B_1),$$

$$2BX \sin\theta = QB \sin(\theta - B_1 + B) + PB \sin(\theta - B_1),$$

$$2ZB \sin\theta = QB \sin(\theta - B_1) + PB \sin(\theta - B_1 + B).$$

But $$OX \sin B = ZB - BX \cos B.$$

Hence $$\begin{aligned} 2OX \sin B \sin\theta &= QB \sin(\theta - B_1) + PB \sin(\theta - B_1 + B) \\ &\quad - QB \sin(\theta - B_1 + B) \cos B \\ &\quad - PB \sin(\theta - B_1) \cos B \\ &= \sin B\{ - QB \cos(\theta - B_1 + B) + PB \cos(\theta - B_1)\}. \end{aligned}$$

Thus $$2OX \sin\theta = - QB \cos(\theta - B_1 + B) + PB \cos(\theta - B_1).$$

Now $$\rho^2 = DX^2 + OX^2,$$

and so $$4\rho^2 \sin^2\theta = PB^2 + QB^2 - 2PB \cdot QB \cos B, \quad \text{.....(iv)}$$

$$4\rho^2 \sin^2\theta = PC^2 + QC^2 - 2PC \cdot QC \cos C, \quad \text{.....(v)}$$

$$4\rho^2 \sin^2\theta = PA^2 + QA^2 - 2 \cdot PA \cdot QA \cos A. \quad \text{.....(vi)}$$

4. *Angles of intersection.*

$$\begin{aligned} \angle FDE &= (B - B_1) + C_1 = \angle CPB - A, \\ \angle DEF &= (C - C_1) + A_1 = \angle APC - B, \\ \angle EFD &= (A - A_1) + B_1 = \angle BPA - C, \\ \angle NLM &= (C - C_1) + B_1 = \angle CQB - A, \\ \angle LMN &= (A - A_1) + C_1 = \angle AQC - B, \\ \angle MNL &= (B - B_1) + A_1 = \angle BQA - C. \end{aligned}$$

Further, EF intersects QA at angle θ,
FD intersects QB at angle θ,
DE intersects QC at angle θ.

Similarly, MN intersects PA at angle $\pi - \theta$,
NL intersects PB at angle $\pi - \theta$,
LM intersects PC at angle $\pi - \theta$.

5. *The locus of the centre.*

$$PQ^2 = PB^2 + QB^2 - 2PB \,.\, QB \cos(2B_1 - B).$$

Now $$OP^2 = (PB \sin B_1 - OX)^2 + (BX - PB \cos B_1)^2,$$

so that
$$\begin{aligned} 4OP^2 \sin^2\theta &= \{QB \cos(\theta - B_1 + B) - PB \cos(\theta + B_1)\}^2 \\ &\quad + \{QB \sin(\theta - B_1 + B) - PB \sin(\theta + B_1)\}^2 \\ &= PB^2 + QB^2 - 2PB \,.\, QB \cos(2B_1 - B) \\ &= PQ^2. \end{aligned}$$

Hence $$PQ = 2OP \sin\theta, \quad \text{.....(vii)}$$

and similarly $$PQ = 2OQ \sin\theta. \quad \text{.....(viii)}$$

Thus $$OP = OQ, \quad \text{.....(ix)}$$

and hence the locus of the centre of the isogonal circle is the perpendicular bisector of PQ.

Also, from (vii) or (viii), $\angle POQ = 2\theta$. .....(x)

Further, if d is the distance of O from PQ,

$$d = \tfrac{1}{2}PQ \cot\theta. \quad \text{.....(xi)}$$

When $\theta < \frac{1}{2}\pi$, O is as in the diagram; when $\theta > \frac{1}{2}\pi$, O is on the other side of PQ; and when $\theta = \frac{1}{2}\pi$, O is the midpoint of PQ. The circle whose centre is the midpoint of PQ and which passes through the feet of the perpendiculars from P and Q to the sides is the ortho-isogonal or *central* circle of the whole system. If the radius of this central circle be denoted by ρ_c we have

$$4\rho_c^2 = PA^2 + QA^2 - 2PA \,.\, QA \cos A, \text{ etc.};$$

hence the radius of any circle of the system is given by the very simple formula

$$\rho = \rho_c \operatorname{cosec}\theta. \quad \text{.....(xii)}$$

6. *The generating point of the system.* Let DM and FL intersect at U, EN and DM at V, FL and EN at W. Then AU, BV, CW are concurrent.

Proof. Let AU, BV, CW intersect the sides at H, K, T respectively. Then

$$BH/HC = c \,.\, \sin HAB/b \,.\, \sin CAH = c \,.\, FU \,.\, \sin AFU/b \,.\, UM \,.\, \sin UMA.$$

But $FU/UM = FD/LM$, since FUL and DUM are chords, and

$$\sin AFU/\sin UMA = \sin NFL/\sin DME = \sin LMN/\sin EFD = NL/DE.$$

Thus $BH/HC = c \,.\, FD \,.\, NL/b \,.\, DE \,.\, LM = c \,.\, bPB \,.\, b\,QB/b \,.\, cPC \,.\, cQC$;

that is, $$BH/HC = b \,.\, PB \,.\, QB/c \,.\, PC \,.\, QC;$$

also $$CK/KA = c \,.\, PC \,.\, QC/a \,.\, PA \,.\, QA,$$

and $$AT/TB = a \,.\, PA \,.\, QA/b \,.\, PB \,.\, QB.$$

Thus AH, BK, CT are fixed concurrent lines. Thence AU, BV, CW are concurrent at a fixed point, R, say; that is, the vertices of the triangle UVW whose sides intersect BC, CA, AB in six cyclic points, lie on the fixed lines RA, RB, RC. But $CD/CM = PC/QC = \text{constant}$. Thus UV has a fixed direction, and similarly so have VW, WU. Hence whatever value θ may have, the various triangles UVW will be homothetic and the point R will be the homothetic centre.

Hence R is the generating point of the whole system of isogonal circles, and the variable triangle UVW whose sides are always parallel to DM, EN, and FL will generate the circles.

Two results relating to the point R are very important.

(a) PH and QH are equally inclined to BC,

PK and QK are equally inclined to CA,

PT and QT are equally inclined to AB.

Hence a very simple construction will give R.

(b) When the triangle UVW reduces to zero (at the point R) its sides will become three concurrent lines through R in the fixed directions of the lines DM, EN, FL. These concurrent lines through R will give six concyclic points. I shall call the circle concerned the *concurrency circle* of the system.

7. *Applications.* (A) When $A_1 = \frac{1}{2}A$, $B_1 = \frac{1}{2}B$, $C_1 = \frac{1}{2}C$, P and Q coincide at I, the in-centre, and the *central* circle of the isogonal system is the in-circle. Hence

$$\rho_c = r.$$

The lines joining the vertices of the triangle ABC to the feet of the perpendiculars from I to the sides are concurrent in R, the generating point. If through R three lines be drawn making angles $\frac{1}{2}\pi - \frac{1}{2}C$, $\frac{1}{2}\pi - \frac{1}{2}A$, $\frac{1}{2}\pi - \frac{1}{2}B$ with the sides BC, CA, AB respectively, the six points will lie on the concurrency circle. Its radius is r cosec θ, where

$$\tan\theta = \tan\tfrac{1}{2}A + \tan\tfrac{1}{2}B + \tan\tfrac{1}{2}C.$$

(B) When $A_1 = \frac{1}{2}\pi - C$, $B_1 = \frac{1}{2}\pi - A$, $C_1 = \frac{1}{2}\pi - B$, P is the circumcentre and Q the orthocentre. The *central* circle is the nine-points circle, and

$$\rho_c = \tfrac{1}{2}R.$$

The generating point R is the isotomic conjugate of the circumcentre (that is, if AP and AR are produced to intersect BC, they will cut off equal intercepts at the ends of BC: and similarly with BP, BR and with CP, CR on the other sides). The radius of the concurrency circle is

$$\tfrac{1}{2}R \text{ cosec } \theta,$$

where

$$\cot\theta = -(\sin 2A \cos 2B + \sin 2B \cos 2C + \sin 2C \cos 2A)$$
$$\div (\sin 2A \sin 2B + \sin 2B \sin 2C + \sin 2C \sin 2A).$$

Two other circles of this system are interesting. When $\theta = \pi/6$ or $5\pi/6$, $\rho = \rho_c \text{ cosec } \pi/6 = R$; that is, the circles concerned have the same radius as the circumcircle, and their centres are the vertices of equilateral triangles described on the line joining the circumcentre to the orthocentre.

(C) When $A_1 = B_1 = C_1 = \omega$ (the Brocard angle) so that

$$\cot\omega = \cot A + \cot B + \cot C,$$

P and Q are then the Brocard points and the system of circles obtained by

giving different values to θ is known as the Tucker system. It is very important, but, as our theorem shows, it is only one of an infinite number of similar systems. When $\theta = \omega$, the isogonal circle is the circumcircle. When $\theta = \frac{1}{2}\pi$, we obtain the *central* or Brocard-pedal circle for which $\rho_c = R \sin \omega$. When $\theta = 2\omega$, we obtain the concurrency circle. This is Lemoine's circle for which $\rho = \frac{1}{2}R \sec \omega$. The generating point is the symmedian point (that is, the point whose distances from the sides are proportional to the respective sides). When $\theta = \frac{1}{2}\pi + \omega$, we obtain the cosine circle. Here $\rho = R \tan \omega$ and the centre is the symmedian point. When

$$\tan \theta = \sin^2 \omega / (\sin \omega \cos \omega - \sin A \sin B \sin C)$$

the circle obtained is Taylor's circle, for which

$$\rho^2 = R^2 (\sin^2 A \sin^2 B \sin^2 C + \cos^2 A \cos^2 B \cos^2 C).$$

(D) When P is the centroid, Q is the symmedian point. The *central* circle of this system has a very interesting radius, namely,

$$\rho_c = m_1 m_2 m_3 / (m_1^2 + m_2^2 + m_3^2),$$

where m_1, etc., are the medians of the triangle ABC; $BH/HC = m_2^2/m_3^2$, etc., and PH is therefore a symmedian of the triangle PBC.

(E) When P is the equiangular or Fermat point ($\angle BPA = \angle CPB = \angle APC$), Q is the Apollonian point ($a \,.\, QA = b \,.\, QB = c \,.\, QC$). Here

$$\rho_c^2 = 2\Delta/3 \,(\cot \omega + \sqrt{3}),$$

PH bisects the angle CPB, etc. Hence the generating point is readily obtained. When $\theta = \frac{1}{3}\pi$, the isogonal circle is the concurrency circle. If perpendiculars be drawn from Q to the sides, the feet will form an equilateral triangle. The locus of the centres of the circumcircles of all equilateral triangles inscribed in the triangle ABC is the perpendicular bisector of PQ.

N. Y. W.

1652. I never appreciated arithmetical progression until, in Busto's apartment-house, I learnt how tea three times infused becomes intolerably weak, and how cigarette-ends twice rolled grow unbearably strong. I may have learned a little geometry at school, but I had to struggle with Busto's blankets before I realized how ridiculously incongruous two rectangles can be, and I had to sleep on one of Busto's beds before I got to know the difference between looking at an angle and lying on one.—Gerald Kersh, "Karmesin" (short story in *Argosy Magazine*, December 1949.) [Per Mr. A. R. Pargeter.]

1653. The whole of the process in the Fifth Book [of Euclid] is purely logical, that is, the whole of the results are virtually contained in the definitions, in the manner and sense in which metaphysicians (certain of them) imagine all the results of mathematics to be contained in their definitions and hypotheses. No assumption is made to determine the truth of any consequence of this definition, which takes for granted more about number or magnitude than is necessary to understand the definition itself. The latter being once understood, its results are deduced by inspection—of itself only, without the necessity of looking at anything else. Hence, a great distinction between the fifth and the preceding books presents itself. The first four are a series of propositions, resting on different fundamental assumptions; that is, about different kinds of magnitudes. The fifth is a definition and its development; and if the analogy by which names have been given in the preceding Books had been attended to, the propositions of that Book would have been called *corollaries of the definition*.—A. De Morgan. *Connexion of Number and Magnitude.* [Per Mr. B. A. Swinden.]

MATHEMATICAL NOTES.

2168. *The circular functions for all angles.*

The circular functions for the general angle can be dealt with very simply by defining them as such that the functional relations obtained for acute angles will still hold in general. This is a more logical procedure than the formation of intricate definitions to be followed by more complicated proofs of the addition formulae.

The essential relations for sine and cosine are

$$\sin A = \cos(90^\circ - A), \quad \text{..... (I)}$$

$$\sin(A+B) = \sin A \,.\, \cos B + \cos A \,.\, \sin B. \quad \text{.....(II)}$$

If we accept the values obtained by the usual methods for angles from 0° to 90° inclusive and let (I) and (II) hold for *all angles*, then the argument may be as follows.

In (I) put

$$A = 90^\circ - B \text{ and get } \sin(90^\circ - B) = \cos A, \quad \text{.....(a)}$$

$$\sin(90^\circ + A) = \sin 90^\circ \,.\, \cos A + \cos 90^\circ \,.\, \sin A = \cos A, \quad \text{.....(b)}$$

$$\cos(-A) = \sin(90^\circ - (-A)) = \sin(90^\circ + A) = \cos A, \quad \text{.....(c)}$$

$$0 = \sin 0^\circ = \sin(A + (-A)) = \sin A \,.\, \cos(-A) + \cos A \,.\, \sin(-A)$$
$$= \cos A\,(\sin A + \sin(-A)), \quad \therefore \sin A = -\sin(-A), \quad \text{.....(d)}$$

$$\sin(A-B) = \sin(A + (-B)) = \sin A \,.\, \cos(-B) + \cos A \,.\, \sin(-B)$$
$$= \sin A \,.\, \cos B - \cos A \,.\, \sin B, \quad \text{.....(e)}$$

$$\cos(A+B) = \sin(\overline{90^\circ - A} - B) = \sin(90^\circ - A) \,.\, \cos B - \cos(90^\circ - A) \,.\, \sin B$$
$$= \cos A \,.\, \cos B - \sin A \,.\, \sin B, \quad \text{.....(f)}$$

$$\cos(A-B) = \cos A \,.\, \cos(-B) - \sin A \,.\, \sin(-B)$$
$$= \cos A \,.\, \cos B + \sin A \,.\, \sin B. \quad \text{.....(g)}$$

$$\text{Also } 1 = \cos 0^\circ = \cos(A-A) = \cos^2 A + \sin^2 A. \quad \text{.....(h)}$$

Thus, the Addition and Pythagorean formulae, and all derived formulae, for all the circular functions hold generally.

The Periodic Nature is established in the following sequence :

$$\sin A = \cos(90^\circ - A) = \sin(90^\circ + \overline{90^\circ - A}) = \sin(180^\circ - A),$$

$$\sin(180^\circ + A) = \sin(90^\circ + \overline{90^\circ + A}) = \cos(90^\circ + A) = -\sin A,$$

$$\sin(360^\circ + A) = \sin(180^\circ + \overline{180^\circ + A}) = -\sin(180^\circ + A) = \sin A,$$

whence, if $360^\circ + A = B$ then $\sin B = \sin(B - 360^\circ)$,

$$\cos A = \sin(90^\circ + A) = \sin(360^\circ + \overline{90^\circ + A}) = \sin(90^\circ + \overline{360^\circ + A})$$
$$= \cos(360^\circ + A).$$

It follows that the sine and cosine (and therefore the other four functions) are periodic with a period of 360°.

F. JACKSON.

2169. *The theorem of v. Staudt and Clausen.*

This famous theorem is often proved by methods which obscure its real origin. Nielsen in his *Traité élémentaire des nombres de Bernoulli* gives one without this defect, which he says is " une légère modification de celle de l'illustre géomètre allemand ", v. Staudt. Bachmann also gives v. Staudt's proof in his *Niedere Zahlentheorie*, but prefers the proof by Lucas, for reasons I find unconvincing. The following is a slight modification of Nielsen's presentation. It assumes nothing except the definition of the numbers and the use of moduli.

1. Divide the integers 1, 2, ..., pq into q sets as follows :

$$\begin{array}{llll} 1, & 2, & 3, & \dots, \quad p \\ p+1, & p+2, & p+3, & \dots, \quad 2p \\ 2p+1, & 2p+2, & 2p+3, & \dots, \quad 3p \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ (q-1)p+1, & (q-1)p+2, & (q-1)p+3, & \dots, \quad qp. \end{array}$$

Now $$a^n \equiv (a+bp)^n, \pmod{p}.$$

Hence $$\sum_{r=1}^{pq} r^n \equiv q \cdot \sum_{1}^{p} r^n, \pmod{p},$$

$$\frac{1}{pq}\sum_{r=1}^{pq} r^n - \frac{1}{p}\sum_{1}^{p} r^n = \frac{h}{q},$$

$$\frac{1}{pq}\sum_{r=1}^{pq} r^n - \frac{1}{p}\sum_{1}^{p} r^n - \frac{1}{q}\sum_{1}^{q} r^n = \frac{h'}{q}, \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1)$$

where h, h' and h'' below are integers. Interchanging the roles of p and q, the left-hand side of the last equation must also be of form h''/p.

Now let p, q be co-prime ; then since ph'/q is an integer, so must h'/q be. The left-hand side of (1) is an integer.

2. Generally, if $p_1, p_2, \dots p_s$ be distinct primes, with product P, then

$$\frac{1}{P}\sum_{1}^{P} r^n - \frac{1}{p_1}\sum_{1}^{p_1} r^n - \dots - \frac{1}{p_s}\sum_{1}^{p_s} r^n$$

is an integer. All the sums are taken with respect to r.

3. We have

$$(x+1)^{n+1} - x^{n+1} = \binom{n+1}{1} x^n + \binom{n+1}{2} x^{n-1} + \dots + \binom{n+1}{1} x + 1.$$

Put in succession $x = a-1, a-2, \dots, 0$, and add, then

$$a^{n+1} - a = \binom{n+1}{1}\sum_{r=1}^{a-1} r^n + \binom{n+1}{2}\sum_{r=1}^{a-1} r^{n-1} + \dots + \binom{n+1}{1}\sum_{r=1}^{a-1} r. \quad \dots\dots (2)$$

First, suppose $n+1$ is a prime number p, then since

$\binom{p}{1}, \binom{p}{2}, \dots$ are all divisible by p, we have Fermat's theorem :

$$a^p - a \equiv 0 \pmod{p}, \text{ and, if } a \text{ is prime to } p,$$
$$a^{p-1} - 1 \equiv 0 \pmod{p}.$$

Next, take a to be an odd prime p, then by (2)

$$\binom{n+1}{1}\sum_{r=1}^{p-1} r^n + \binom{n+1}{2}\sum_{r=1}^{p-1} r^{n-1} + \dots + \binom{n+1}{1}\sum_{r=1}^{p-1} r \equiv 0, \pmod{p}. \quad \dots\dots (3)$$

Now $$\sum_{r=1}^{p-1} r = \tfrac{1}{2}p(p-1) \equiv 0 \pmod{p}.$$

Suppose $$\sum_{r=1}^{p-1} r^m \equiv 0 \pmod{p} \text{ for } m = 1, 2, \dots, n-1,$$

and that $n < p-1$, then by (3)

$$\binom{n+1}{1}\sum_{1}^{p-1} r^n \equiv 0 \pmod{p},$$

and since $n<p-1$, we have

$$\sum_{1}^{p-1} r^n \equiv 0 \pmod{p}.$$

If $n \geqslant p-1$, let $n=(p-1)k+l$, $l<p-1$, then

$$r^n = r^l\{r^{p-1}\}^k \equiv r^l \pmod{p},$$

by Fermat's theorem, since r is less than p.

Hence we have for any n,

$$\sum_{r=1}^{p-1} r^n \equiv \sum_{r=1}^{p-1} r^l \equiv 0 \text{ or } -1 \pmod{p}$$

according as l is greater than or equal to zero, that is, according as $p-1$ does not, or does, divide n.

4. Let P be the product of all primes $p_0=2, p_1, p_2, \ldots, p_s$ not greater than $n+1$. Then by § 2

$$\frac{1}{P}\sum_{r=1}^{P} r^n - \tfrac{1}{2}(1+2^n) - \Sigma_i\left(\frac{1}{p_i}\sum_{1}^{p_i} r^n\right)$$

is an integer. Amongst the odd primes $p_1, p_2, \ldots, p_s$, let $\lambda_1, \lambda_2, \ldots$ be those such that $\lambda_i - 1$ divides n. Then by § 3

$$\frac{1}{P}\sum_{1}^{P} r^n + \tfrac{1}{2} + \Sigma\frac{1}{\lambda_i} \text{ is an integer.}$$

It is this fact which is the real source of the *v. Staudt-Clausen theorem, which states that*

$$B_{2m} = \text{Integer} - \tfrac{1}{2} - \Sigma(1/\lambda_i),$$

where λ_i traverses the odd primes such that $\lambda_i - 1$ divides $2m$. (The integer may be negative.)

5. Assume the theorem known for $B_2, B_4, \ldots, B_{2m-2}$. Let P be the product of all primes not greater than $2m+1$. Then B_2P, B_4P, $\ldots$ $B_{2m-2}P$ are integers.

Now

$$\begin{aligned}\frac{1}{P}\sum_{1}^{P} r^{2m} = \frac{P^{2m}}{2m+1} &+ \tfrac{1}{2}P^{2m-1} + 2m\,\frac{B_2}{2!}P^{2m-2} \\ &+ 2m(2m-1)(2m-2)\,\frac{B_4}{4!}P^{2m-4} + \ldots \\ &+ 2m(2m-1)\ldots 4\cdot\frac{B_{2m-2}}{(2m-2)!}P^2 + B_{2m}.\end{aligned}$$

All terms on the right-hand side, except B_{2m}, will be proved to be integers if we show that

$$\frac{P^{2m-1}}{2m+1}, \quad \binom{2m}{2}\frac{P^{2m-3}}{2m-1}, \quad \binom{2m}{4}\frac{P^{2m-5}}{2m-3}, \quad \ldots, \quad \binom{2m}{2m-2}\frac{P}{3}$$

are integers, and this is done if we prove that $k+2$ divides P^k for $k=2, 3, \ldots, 2m-1$. Suppose p^α is the highest power of the prime $p(\leqslant 2m+1)$ which divides $k+2$. Then p divides P, p^k divides P^k;

$$p^k \geqslant 2^k \geqslant k+2 \geqslant p^\alpha.$$

Hence $k \geqslant \alpha$, p^α divides P^k. Hence $k+2$ divides P^k.

Thus, finally,

$$\frac{1}{P}\sum_{1}^{P} r^{2m} - B_{2m}$$

is an integer, and this, with the result of 4, with $n=2m$, proves the theorem.

5. For example, $B_2 = 1 - (\frac{1}{2} + \frac{1}{3})$, $B_{10} = 1 - (\frac{1}{2} + \frac{1}{3} + \frac{1}{11})$,

$$B_{16} = -6 - (\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{17}).$$

If we call the sum of B_{2m} and the corresponding v. Staudt fractions the *integral part* of B_{2m}, calculation shows that the integral parts of B_2, B_4, ... , B_{14} are all positive. As B_{2m} when m is odd is positive this is to be expected for these numbers, but we can show that when m is *even* then B_{16}, B_{20}, ... and all such succeeding have negative integral parts. For the formula connecting B_{4m}, S_{4m}, using Stirling, gives

$$-B_{4m} = \frac{2(4m)!}{(2\pi)^{4m}} S_{4m} > 4\pi\sqrt{e}\left(\frac{2m}{\pi e}\right)^{4m+\frac{1}{2}},$$

whereas the fractional part is

$$\tfrac{1}{2} + \Sigma(1/\lambda_i) \leqslant \tfrac{1}{2} + \tfrac{1}{3}(2m-1) = \tfrac{1}{6}(4m+1),$$

which is less than the previous expression when $m \geqslant 4$.

I have to thank Mr. B. I. Hayman for checking this note. H. G. FORDER.

2170. *The use of vectors for finding OH^2, OI^2.*

1. Take any fixed origin and denote the vector from the origin to a point P by the corresponding small letter p. Thus when we consider a triangle ABC, the symbols a, b, c represent vectors to A, B, C, and we shall have to denote the lengths of the sides of the triangle by other letters, say α, β, γ.

If we take the origin at the circumcentre, then

$$a^2 = b^2 = c^2 = R^2,$$

where R is the circumradius. The vector to the centroid G is always

$$s = (a + b + c)/3.$$

Take H so that $OH = 3OG$, then $h = a + b + c$, and, using scalar products,

$$(h - a)(b - c) = (b + c)(b - c) = b^2 - c^2 = 0.$$

Thus HA is perpendicular to BC, and so on. Hence H is the orthocentre.

$$\begin{aligned} OH^2 = h^2 = (a+b+c)^2 &= a^2 + b^2 + c^2 + 2(bc + ca + ab) \\ &= 3R^2 + 2R^2(\cos 2A + \cos 2B + \cos 2C) \\ &= R^2(1 - 8\cos A \cos B \cos C), \end{aligned}$$

since $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1$.

2. The incentre I is the centre of gravity of weights α, β, γ at A, B, C. Hence the vector OI is

$$(\alpha a + \beta b + \gamma c)/2s, \quad \text{where} \quad 2s = \alpha + \beta + \gamma.$$

$$\begin{aligned} 4s^2 \,.\, OI^2 = (\alpha a + \beta b + \gamma c)^2 &= \alpha^2 a^2 + \beta^2 b^2 + \gamma^2 c^2 + 2\beta\gamma bc + \ldots + \ldots, \\ &= R^2(\alpha^2 + \beta^2 + \gamma^2 + 2\beta\gamma \cos 2A + \ldots + \ldots) \\ &= R^2\{(\alpha+\beta+\gamma)^2 - 4(\beta\gamma \sin^2 A + \ldots + \ldots)\} \\ &= R^2\{4s^2 - 4 \,.\, \Delta \,.\, (\alpha+\beta+\gamma)/R\} \\ &= 4s^2(R^2 - 2R\Delta/s). \end{aligned}$$

Thus $OI^2 = R^2 - 2Rr$. H. G. FORDER.

2171. *A problem in solid geometry.*

Recently one of those problems arose which is easily stated, but the solution of which appears to be less straightforward than might be expected, although

it does not imply anything beyond elementary theorems of plane and, in one respect, of spherical trigonometry. It is required to find the volume of that portion of a sphere, which is cut out by two planes, such that only one of them passes through the centre of the sphere. (When both of them pass through the centre, the problem becomes trivial, but this special case might form a useful check on the final answer.)

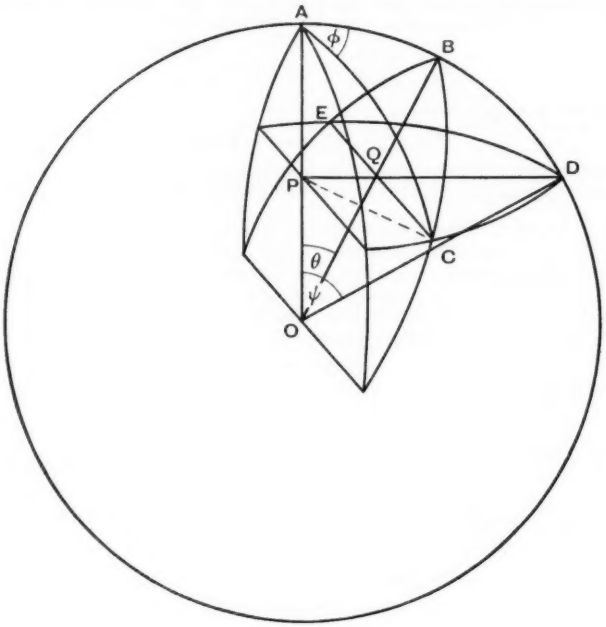


FIG. 1.

Let the two planes in question be CBE (α) and CDE (β) (see diagram 1), the first passing through the centre O. The radius of the sphere will be assumed to be unity. Since the volume required is symmetrical about a plane γ perpendicular to both α and β, the problem can be reduced to finding the volume cut out by the three planes α, β and γ. This volume will be called $V/2$, V being the volume to be found in the problem as set originally.

The three planes cut the surface of the sphere to form the (spherical) triangle BCD.

To find the value $V/2$, we shall first find the volume V_1, say, of the solid formed by joining O to the sides BC, CD and DB, and subtracting from it the volume V_2, say, of the cone with apex O having as base the area QCD in plane β.

If the surface area BCD is called F, it is easily seen that the volume V_1 equals $F/3$. Hence our first task is to find F. Of the sides of this area, two are portions of great circles, while the third, viz. CD, is not. We note, however, that F is the difference of the areas of the sector ACD of the cap cut

out by β and the spherical triangle ABC, of which all the sides are portions of great circles. Let ϕ be the angle between the plane AOC and γ; then $\cos\phi = \tan\theta \cot\psi$.

Now the area of the curved surface of the complete cap cut off by β is

$$2\pi(1 - \cos\psi)$$

(by the device of projecting on to a circular cylinder having the same diameter). The radius of the base of this cap is $\sin\psi$, and hence the length of

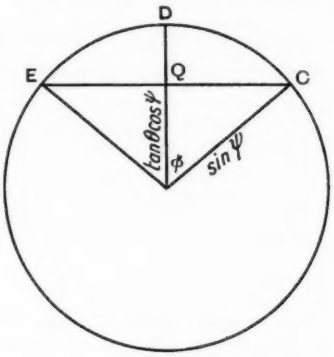


FIG. 2.

arc CD is $\phi \sin\psi$ (see diagram 2). It follows that the area of the sector ACD is

$$\phi \sin\psi \,.\, 2\pi(1 - \cos\psi)/2\pi \sin\psi = \phi(1 - \cos\psi).$$

As mentioned above, we have to subtract from this the area of the spherical triangle ABC (diagram 3). We have $\sin C = \sin\theta/\sin\psi$, and therefore the

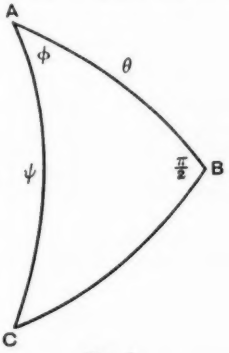


FIG. 3.

area of ABC, which equals the spherical excess in radians, is

$$\frac{\pi}{2} + \phi + \sin^{-1}\left(\frac{\sin\theta}{\sin\psi}\right) - \pi.$$

Thus we have found, for the area F of BCD, as the difference of the areas of ACD and ABC

$$(1-\cos\psi)\,\phi+\frac{\pi}{2}-\phi-\sin^{-1}\frac{\sin\theta}{\sin\psi}=F.$$

The next step is the calculation of the volume V_2 of the cone $OQCD$. The area of the base QCD is easily seen, from Diagram 2, to be

$$\tfrac{1}{2}[\phi\sin^2\psi-\tan^2\theta\cos^2\psi\tan\phi],$$

and hence the volume $V_2=\dfrac{\cos\psi}{6}[\phi\sin^2\psi-\tan^2\theta\cos^2\psi\tan\phi]$.

The volume of the portion of the sphere which we wanted to find is thus, finally, $V=2F/3-2V_2$, *i.e.*

$$\tfrac{1}{3}\left[-2\phi\cos\psi+\pi-2\sin^{-1}\frac{\sin\theta}{\sin\psi}-\phi\cos\psi\sin^2\psi+\tan^2\theta\cos^3\psi\tan\phi\right].$$

The more general problem in which neither plane passes through the centre of the sphere can be solved by reducing it to the special case considered above, by dividing the volume into two parts by a plane through the intersection of the two given planes and the centre of the sphere. A. P. Bailey.

2172. *One-one correspondence.*

The necessary conditions for a homographic correspondence between two plane figures are not often stated. In Filon's *Projective Geometry* they are given as (i) the relation between the points must be algebraic, (ii) it must be one-one, and (iii) a straight line must correspond to a straight line. This last condition (iii) is, however, unnecessary provided the planes are treated projectively and have lines at infinity, and (i), (ii) apply to all points, including those at infinity. The condition (iii) was no doubt introduced to exclude other one-one correspondences, such as those in the complex or inversion planes.

The following is suggested as a simple proof.

If $P(x, y, z)$, $P'(x', y', z')$ are corresponding points, the relation between their coordinates, if one-one and algebraic, must be given by simultaneous linear equations in either $x : y : z$ or $x' : y' : z'$, and thus must be

$$u_1x'+u_2y'+u_3z'=0,$$
$$v_1x'+v_2y'+v_3z'=0,$$

where the u's and v's are linear in x, y, z. These give $x' : y' : z'$ uniquely unless $u_1/v_1=u_2/v_2=u_3/v_3$ simultaneously.

If no u or v is identically zero, these equations are satisfied by at least one set of values of $x : y : z$, say a common point of the conics $u_1v_2-u_2v_1=0$, $u_2v_3-u_3v_2=0$, and to this point would correspond any point on either of the lines

$$u_1x'+u_2y'+u_3z'=0,\quad v_1x'+v_2y'+v_3z'=0,$$

and the relation would not be one-one. Hence at least one u or v is identically zero, say $v_3\equiv 0$. Then

$$x' : y' : z' = -u_3v_2 : u_3v_1 : u_1v_2-u_2v_1.$$

Here if u_3 is not a factor of $u_1v_2-u_2v_1$, to any point on the line $u_3=0$ corresponds the single point $(0, 0, 1)$ for x', y', z', and again the relation is not one-one. Thus u_3 must be a factor of $u_1v_2-u_2v_1$, so that

$$u_1v_2-u_2v_1\equiv u_3w,$$

where w is linear in x, y, z.

Hence

$$x' : y' : z' = (a_1x + b_1y + c_1z) : (a_2x + b_2y + c_2z) : (a_3x + b_3y + c_3z)$$

is necessary for algebraic one-one correspondence, and by solving for $x : y : z$, it is found to be sufficient provided the determinant of the coefficient is not zero.

This proof does not take into account the difficult question as to whether any of the points may be regarded as repeated, which is considered in J. A. Todd's *Projective Geometry*, Ch. II. The necessity for this, however, can be avoided by ensuring that the correspondence at each stage is algebraically one-one. A plausible example is the correspondence between any point P on a fixed tangent to a conic and P', the point of contact of the second tangent from P. It might be supposed that to each value of P corresponds two coincident values of P', but actually P' is not given as the coincident points of intersection of a chord through P, but as the point of contact of the second tangent from P, which is unique. H. V. MALLISON.

2173. *Three converses.*

If A, B, C, P are four collinear points, the three relations between directed segments

$$BC + CA + AB = 0, \quad \text{.....(i)}$$

$$AP \cdot BC + BP \cdot CA + CP \cdot AB = 0, \quad \text{.....(ii)}$$

$$AP^2 \cdot BC + BP^2 \cdot CA + CP^2 \cdot AB = -BC \cdot CA \cdot AB \quad \text{.....(iii)}$$

are illustrations of three cognate algebraic identities. Their converses, however, present some different points of interest.

It may be noted that (iii) is also true if P is any point of space. Also if A, B, C are any points of a plane or in space subject to one of the above conditions, one of BC, CA, AB must be considered to have the sign opposite to that of the other two. Suppose $BC = a$, $AC = -CA = b$, $AB = c$, where a, b, c are positive.

In (i), if $b = a + c$, then A, B, C must be collinear.

In (ii), if $BP \cdot AC = AP \cdot BC + CP \cdot AB$, in general P lies on the circle ABC, by the converse of Ptolemy's theorem. It is noteworthy that although there is an additional relation between the six mutual distances of the four points if they are coplanar (H. F. Baker, *Introduction to Plane Geometry*, p. 365, Ex. 5) the condition (ii) alone is sufficient, provided the points are real, to ensure that the points are concyclic and therefore coplanar. This is easily seen by inversion with respect to P.

In (iii) if G is the mean centre for multiples a, $-b$, c at A, B, C, then

$$a \cdot AP^2 - b \cdot BP^2 + c \cdot CP^2 = a \cdot AG^2 - b \cdot BG^2 + c \cdot CG^2 + (a - b + c)PG^2.$$

Here G is I_2, the centre of the escribed circle touching CA internally, and

$$a \cdot AI_2^2 - b \cdot BI_2^2 + c \cdot CI_2^2 = -abc,$$

for if I_2X, I_2Y, I_2Z are perpendiculars to BC, CA, AB,

$$\text{quad. } I_2ZBX - \text{quad. } I_2YCX - \text{quad. } I_2ZAY = \text{triangle } ABC.$$

If H is the image of I_2 in BC,

$$\text{quad. } I_2ZBX = 2\Delta I_2BX = \Delta I_2BH,$$

and

$$\Delta I_2BH : \Delta ABC = I_2B^2 : BA \cdot BC.$$

A similar method applied to the other quadrilaterals gives the result. Hence the converse of (iii) is

$$(a-b+c)\,I_2P^2 = 2abc$$

or

$$I_2P^2 = abc/(s-b) = 4Rr_2.$$

Thus taking the signs as shown, P lies on the sphere with centre I_2 and radius $2\sqrt{(Rr_2)}$.

When $a+c-b$ tends to zero, this sphere tends to the plane through the line ABC perpendicular to the plane ABC, which then becomes indeterminate. Hence when A, B, C are collinear, (iii) is true for any point in space.

H. V. MALLISON.

2174. *Rational common tangents.*

Let R_1, R_2 $(R_2 > R_1)$ be the radii of two circles, whose centres are at a distance D apart; then the lengths of the direct and transverse common tangents are

$$T_1 = \sqrt{\{D^2 - (R_1 - R_2)^2\}},$$

$$T_2 = \sqrt{\{D^2 - (R_1 + R_2)^2\}}.$$

For the purpose of instruction in the method of constructing the common tangents, and calculating their lengths, it would be convenient to have a set of rational values of D, R_1, R_2, T_1 and T_2.

Such sets of values are not difficult to find, but practical considerations require that the ratio $R_1 : R_2$ should not be too close to zero or unity, and that the extreme length of the figure, namely, $R_1 + R_2 + D$, in inches or centimetres, should be within the limits 6 in. (or 15 cm.) and 10 in. (or 25 cm.), say. I propose the values

$$R_1 = 2, \quad R_2 = 5{\cdot}5, \quad D = 12{\cdot}5, \quad T_1 = 12, \quad T_2 = 10$$

in centimetres.

Can any reader suggest more convenient values? Is there, for example, a solution in integers within the limits proposed? I submit the problem not because it has any innate importance, but because its investigation gave me some amusement which other readers may care to share.

G. TYSON.

2175. *Algebra and the child.*

The following note was suggested by Mr. Gattegno's article entitled "Mathematics and the Child", which appeared in the *Gazette* of May 1949.

One difficulty in teaching children Addition and Subtraction in Algebra is due to the fact that the names of these operations have different meanings in Algebra from those they have in Arithmetic. "Add $+5$ and -2" in Algebra becomes in Arithmetic "Subtract 2 from 5"; and "Subtract -7 from $+2$" has no counterpart in Arithmetic.

Some of the textbook methods of introducing these operations are cumbrous: those depending on debts and credits assume a fuller acquaintance with debts than most children possess; while those which illustrate by means of graph-paper or of imaginary ladders, and the like, tend rather to confuse than to illuminate.

In practice I have found the methods below very effective.

(1) $+1$ added to $-1 = 0$, *i.e.* $+1$ "blots out" -1, and *vice versa*.

(2) Add $+4$ and -6. The work may be set out in this way:

$$+4 = +1 + 1 + 1 + 1$$

$$-6 = -1 - 1 - 1 - 1 - 1 - 1. \quad \text{Ans. } -2.$$

Four pieces of paper marked $+1$, and six marked -1 may be helpful; cover each $+1$ with a -1, and thus leave two pieces marked -1 uncovered.

(3) Before beginning subtraction show that the difference between two numbers is unaltered by increasing or diminishing each number by the same amount. This may be demonstrated by means of two groups of scholars.

(4) What must be added to $+7$ to give -9? ; or, Subtract $+7$ from -9.

-9 Add -7 to each number, and we have $-9-7=-16$,

$+7$ $+7-7=\quad 0$. Ans. -16.

Naturally, I adopt Mr. Gattegno's request, " I would ask the reader not to criticise these methods on philosophical grounds, for I do not here defend my ideas on these grounds."

JAS. W. STEWART.

2176. *On the vector triple product formula.*

Miss H. Perfect of University College, Swansea, has kindly pointed out that my recent proof of the formula *

$$(\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{c}=-\mathbf{a}(\mathbf{b}\,.\,\mathbf{c})+\mathbf{b}(\mathbf{c}\,.\,\mathbf{a})$$

is invalid, because I assumed implicitly, and without justification, that a particular parameter which I called λ was independent of the plane angles between $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively. I have, therefore, revised my proof as follows.

Lemma 1. $(\mathbf{a}\wedge\mathbf{b})^2=\mathbf{a}^2\mathbf{b}^2-(\mathbf{a}\,.\,\mathbf{b})^2=\begin{vmatrix}\mathbf{a}\,.\,\mathbf{a}, & \mathbf{a}\,.\,\mathbf{b}\\ \mathbf{b}\,.\,\mathbf{a}, & \mathbf{b}\,.\,\mathbf{b}\end{vmatrix}$.

This follows immediately from the definitions of $\mathbf{a}\wedge\mathbf{b}$ and $\mathbf{a}\,.\,\mathbf{b}$.

Lemma 2. $(\mathbf{a}\wedge\mathbf{b}\,.\,\mathbf{c})^2=\begin{vmatrix}\mathbf{a}\,.\,\mathbf{a}, & \mathbf{a}\,.\,\mathbf{b}, & \mathbf{a}\,.\,\mathbf{c}\\ \mathbf{b}\,.\,\mathbf{a}, & \mathbf{b}\,.\,\mathbf{b}, & \mathbf{b}\,.\,\mathbf{c}\\ \mathbf{c}\,.\,\mathbf{a}, & \mathbf{c}\,.\,\mathbf{b}, & \mathbf{c}\,.\,\mathbf{c}\end{vmatrix}$.

This determinant will be denoted by the symbol D. To establish Lemma 2, first consider the case in which $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-coplanar. In this case, scalers λ, μ and ν can be found so that

$$\lambda\mathbf{a}+\mu\mathbf{b}+\nu\mathbf{c}=\mathbf{a}\wedge\mathbf{b}. \qquad \text{.....(1)}$$

Multiply equation (1) scalarly by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in turn. Then

$$\left.\begin{aligned}\lambda\mathbf{a}\,.\,\mathbf{a}+\mu\mathbf{a}\,.\,\mathbf{b}+\nu\mathbf{a}\,.\,\mathbf{c}&=0,\\ \lambda\mathbf{b}\,.\,\mathbf{a}+\mu\mathbf{b}\,.\,\mathbf{b}+\nu\mathbf{b}\,.\,\mathbf{c}&=0,\\ \lambda\mathbf{c}\,.\,\mathbf{a}+\mu\mathbf{c}\,.\,\mathbf{b}+\nu\mathbf{c}\,.\,\mathbf{c}&=(\mathbf{a}\wedge\mathbf{b}\,.\,\mathbf{c})\end{aligned}\right\} \qquad \text{.....(2)}$$

Solving equations (2) for ν, we find that

$$\nu=(\mathbf{a}\wedge\mathbf{b}\,.\,\mathbf{c})\{\mathbf{a}^2\mathbf{b}^2-(\mathbf{a}\,.\,\mathbf{b})^2\}/D,$$

whence, by Lemma 1,

$$\nu=(\mathbf{a}\wedge\mathbf{b}\,.\,\mathbf{c})(\mathbf{a}\wedge\mathbf{b})^2/D. \qquad \text{.....(3)}$$

Multiplying equation (1), however, scalarly by $\mathbf{a}\wedge\mathbf{b}$, we see that

$$\nu=(\mathbf{a}\wedge\mathbf{b})^2/(\mathbf{a}\wedge\mathbf{b}\,.\,\mathbf{c}). \qquad \text{.....(4)}$$

Comparing (3) and (4), we find that

$$(\mathbf{a}\wedge\mathbf{b}\,.\,\mathbf{c})^2=D. \qquad \text{.....(5)}$$

This identity remains true when $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar, since using equations (2), we find that D is then zero, as well as $\mathbf{a}\wedge\mathbf{b}\,.\,\mathbf{c}$.

* *Math. Gaz.*, Vol XXXIII (1949), page 126, Note 2051.

Corollary. Interpreted geometrically, Lemma 2 signifies that the volume V of the tetrahedron formed by three concurrent lines of lengths a, b and c, the plane angles between them being α, β and γ, respectively, is given by

$$36V^2 = a^2b^2c^2(1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma).$$

Lemma 3. $\{(\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{c}\}^2 = \{\mathbf{a}(\mathbf{b}\,.\,\mathbf{c}) - \mathbf{b}(\mathbf{c}\,.\,\mathbf{a})\}^2$.

To prove this result, apply Lemma 1 to the two vectors $\mathbf{a}\wedge\mathbf{b}$ and $\mathbf{c}$. Then,

$$\begin{aligned}\{(\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{c}\}^2 &= (\mathbf{a}\wedge\mathbf{b})^2\mathbf{c}^2 - (\mathbf{a}\wedge\mathbf{b}\,.\,\mathbf{c})^2\\ &= \{\mathbf{a}^2\mathbf{b}^2 - (\mathbf{a}\,.\,\mathbf{b})^2\}\mathbf{c}^2 - D\\ &= \mathbf{a}^2(\mathbf{b}\,.\,\mathbf{c})^2 - 2(\mathbf{a}\,.\,\mathbf{b})(\mathbf{b}\,.\,\mathbf{c})(\mathbf{c}\,.\,\mathbf{a}) + \mathbf{b}^2(\mathbf{c}\,.\,\mathbf{a})^2,\end{aligned}$$

using the usual formula for the expansion of a three-rowed determinant. The result follows.

Theorem. $(\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{c} = -\mathbf{a}(\mathbf{b}\,.\,\mathbf{c}) + \mathbf{b}(\mathbf{c}\,.\,\mathbf{a})$.

To prove this theorem, we first observe that $(\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{c}$ must be coplanar with $\mathbf{a}$ and $\mathbf{b}$, and hence we deduce in the usual way that

$$(\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{c} = \kappa\{\mathbf{a}(\mathbf{b}\,.\,\mathbf{c}) - \mathbf{b}(\mathbf{c}\,.\,\mathbf{a})\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

where κ is a scalar. Comparing (6) with Lemma 3, we immediately see that $\kappa^2 = 1$. The theorem then follows from consideration of the *sense* of vector products, as usually defined.

G. J. WHITROW.

2177. *The triple vector product.*

The following is a proof of the triple vector product formula involving only elementary geometry and a minimum of calculation.

From the fact that $\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})$ is in the plane of $\mathbf{b}$ and $\mathbf{c}$ and is orthogonal to $\mathbf{a}$, we find that

$$\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}) = \lambda[(\mathbf{a}\,.\,\mathbf{c})\mathbf{b} - (\mathbf{a}\,.\,\mathbf{b})\mathbf{c}] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where λ is a scalar, which we must show to be a constant and equal to 1.

We use the fact that the left-hand expression and the expression in brackets on the right of (1) are both linear and homogeneous in all three vectors. If $\mathbf{a}$ is expressed as the sum of its components perpendicular to and in the plane of $\mathbf{b}$ and $\mathbf{c}$, the component of $\mathbf{a}$ perpendicular to the plane contributes nothing to either side of (1), and hence $\mathbf{a}$ may be replaced by its component in the plane of $\mathbf{b}$, $\mathbf{c}$ without altering λ. Moreover, the component of $\mathbf{c}$ parallel to $\mathbf{b}$ contributes nothing to either expression; hence $\mathbf{c}$ may be replaced by its component perpendicular to $\mathbf{b}$ without altering λ; and, in addition, both $\mathbf{b}$ and $\mathbf{c}$ may be replaced by unit vectors in their respective directions without altering λ.

These considerations show that it is sufficient to prove that $\lambda = 1$ in the case where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar and $\mathbf{b}$ and $\mathbf{c}$ are orthogonal and are unit vectors. In this case, $\mathbf{b}\wedge\mathbf{c}$ is the unit vector orthogonal to the plane of $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{a}$ such that rotation from $\mathbf{b}$ to $\mathbf{c}$ is positive viewed from its side of the plane: and $\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})$ is the vector obtained from $\mathbf{a}$ by rotating it through a right-angle in the direction opposite to the rotation from $\mathbf{b}$ to $\mathbf{c}$. It is then obvious that this vector is given by the bracket on the right of (2), so proving the result.

J. L. B. COOPER.

2178. *A note on Whittaker's solution of an algebraic equation.*

Daniel Bernoulli's method of approximating the numerically greatest root of an algebraic equation

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

consists in evaluating $\lim_{t\to\infty} f(t+1)/f(t)$, where

$$a_0 f(t) + a_1 f(t+1) + a_2 f(t+2) + \ldots + a_n f(t+n) = 0$$

gives the values of $f(t)$ when arbitrary values have been assigned to the first n values $f(0), f(1), f(2), \ldots f(n-1)$. A slight modification of this leads at once to Whittaker's * determinantal solution, namely

$$-w = \lim_{n\to\infty} a_0 \begin{vmatrix} a_1 & a_0 & . & \ldots \\ a_2 & a_1 & a_0 & \ldots \\ a_3 & a_2 & a_1 & a_0 \ldots \\ . & . & . & \ldots \\ . & . & . & \ldots \end{vmatrix}_n \div \begin{vmatrix} a_1 & a_0 & . & \ldots \\ a_2 & a_1 & a_0 & \ldots \\ a_3 & a_2 & a_1 & a_0 \ldots \\ . & . & . & \ldots \\ . & . & . & \ldots \end{vmatrix}_{n+1} \quad \ldots\ldots (2)$$

or its Schweinsian expansion

$$\frac{a_0}{a_1}\left\{1 + \frac{a_0 a_2}{\begin{vmatrix} a_1 & a_0 \\ a_2 & a_1 \end{vmatrix}} + a_1 \frac{\begin{vmatrix} a_0 & . \\ a_1 & a_0 \end{vmatrix}\begin{vmatrix} a_2 & a_1 \\ a_3 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_0 \\ a_2 & a_1 \end{vmatrix}\begin{vmatrix} a_1 & a_0 & . \\ a_2 & a_1 & a_0 \\ a_3 & a_2 & a_1 \end{vmatrix}} + \ldots\ldots\ldots\ldots\right\} \quad \ldots (3)$$

where w is the numerically smallest root of (1) and $a_0 \neq 0$. For $1/w$ is the numerically greatest root of

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

for which we consider the difference equation

$$a_0 f(t+n) + a_1 f(t+n-1) + \ldots + a_n f(t) = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

and the limiting value of $f(t+1)/f(t)$.

Now let $f(0)$ take any value and determine $f(1), f(2), f(3), \ldots$, using (5) thus:

$$\begin{aligned} a_0 f(1) &= -a_1 f(0), \\ a_1 f(1) + a_0 f(2) &= -a_2 f(0), \\ a_2 f(1) + a_1 f(2) + a_0 f(3) &= -a_3 f(0), \\ &\vdots \\ a_{n-1} f(1) + a_{n-2} f(2) + \ldots + a_0 f(n) &= -a_n f(0). \end{aligned}$$

Solving for $f(n)$ and transferring the last column, we have

$$\frac{f(n)}{f(0)} = \frac{(-)^n}{{a_0}^n} \begin{vmatrix} a_1 & a_0 & . & . \\ a_2 & a_1 & a_0 & . \\ a_3 & a_2 & a_1 & a_0 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{vmatrix}_n = \frac{(-)^n}{{a_0}^n} A_n, \text{ say,}$$

so that

$$\begin{aligned} w &= \lim f(t)/f(t+1) \\ &= -\lim_{n\to\infty} a_0 \frac{A_n}{A_{n+1}}, \end{aligned}$$

which is Whittaker's solution.

The usual proof uses properties of the homogeneous symmetrical functions of the roots, and does not show the close connection with Bernoulli's method.

L. R. SHENTON.

* Whittaker, *Proc. Edin. Math. Soc.*, 36 (1918).

2179. *A geometric proof of an algebraic identity.*

The identity

$$4(ax+by+cz)^3-3(ax+by+cz)(a^2+b^2+c^2)(x^2+y^2+z^2)-54abcxyz$$
$$=2(b-c)(c-a)(a-b)(y-z)(z-x)(x-y), \quad \ldots\ldots\ldots\ldots(1)$$

valid whenever $x+y+z=a+b+c=0$ (see Note 2045 in *Mathematical Gazette*, Vol. XXXIII, pp. 37-38), can be proved almost without calculation if a, b, c are regarded as arbitrary constants satisfying $a+b+c=0$ and x, y, z as homogeneous co-ordinates of a point. Then (1) represents a cubic, and the proposition to be proved means that this curve contains the line $x+y+z=0$. This follows from the fact that the cubic passes through the four points

$$A(0,\ 1,\ -1),\quad B(1,\ -1,\ 0),\quad C(-1,\ 0,\ 1),\quad D(b-c,\ c-a,\ a-b)$$

of the aforesaid straight line.

Indeed, if in (1) we substitute the co-ordinates of A and if we remove the factor $2(b-c)$, we find $2(b-c)^2-3(a^2+b^2+c^2)=2(c-a)(a-b)$, which becomes an identity if we take $c=-(a+b)$. Similarly, we see that (1) passes through B and C.

If for x, y, z we substitute the co-ordinates of D, we obtain $ax+by+cz=0$ and, as a consequence of $a+b+c=0$, we have $y-z=-3a$, $z-x=-3b$, $x-y=-3c$, so that (1) is again satisfied.

S. K. ZAREMBA.

2180. *Note on Spearman's rank-correlation coefficient* ρ.

It is well known that when the ranks are not duplicated

$$\rho=1-6\sum_{r=1}^{n}(x_r-y_r)^2/(n^3-n)$$

where x_r ; y_r are the rth ranks.

It follows that the necessary and sufficient condition that $\rho=1$ is that the corresponding ranks are equal.

It has not been realised that there is the simple formula

$$\rho=-1+6\sum_{r=1}^{n}\{x_r+y_r-(n+1)\}^2/(n^3-n) \quad \ldots\ldots\ldots\ldots(A)$$

from which one could deduce *that the necessary and sufficient condition that* $\rho=-1$ *is that the corresponding ranks should add to* $n+1$.

No textbook seems to mention such a formula. In Yule and Kendall's *Introduction to the Theory of Statistics*, 13th edition, pp. 248-9, and in Kendall's *Advanced Theory of Statistics*, 2nd edition, Vol. I, p. 389, it is proved that when the ranks are in perfect discordance $\rho=-1$. It is not clear from either reference how the converse could be deduced.

Formula (A) above is easily proved. For

$$\Sigma\{x_r+y_r-(n+1)\}^2=\Sigma\{x_r-\tfrac{1}{2}(n+1)\}^2+\Sigma\{y_r-\tfrac{1}{2}(n+1)\}^2$$
$$+2\Sigma\{x_r-\tfrac{1}{2}(n+1)\}\{y_r-\tfrac{1}{2}(n+1)\}$$
$$=n\{\sigma_x{}^2+\sigma_y{}^2+2\rho\sigma_x\sigma_y\}$$
$$=\tfrac{1}{6}n(n^2-1)(1+\rho)$$

since $\sigma_x{}^2=\sigma_y{}^2=(n^2-1)/12$.

T. TOTADRI IYENGAR.

2181. *Infinite oscillation of alternating series.*

On consulting modern books on advanced algebra and analysis I have not found one which states or proves a single theorem relating to criteria for the

infinite oscillation of an alternating series. Theorems relating to convergence, divergence and finite oscillation of alternating series are given, but there appears to be no statement of general criteria for infinite oscillation. Durell's *Advanced Algebra*, Vol. I, shows in the chapter on series how infinite oscillation may in certain cases be demonstrated by *ad hoc* methods, but does not treat the problem generally.

The following theorem provides a general criterion for infinite oscillation. I have seen no statement or proof of it elsewhere, and should be glad to know if it exists in print in any similar form. It would be suitable subject-matter for scholarship students who, by its aid, would save themselves the trouble of reverting to lengthy *ad hoc* methods and would only need to apply a test quite as simple, for example, as d'Alembert's ratio test for convergence.

Theorem. *If* $A=\sum_1^\infty(-1)^{r-1}a_r$, *where* $a_r>0$ *and* $a_{r+1}/a_r\to k>1$ *as* $r\to\infty$, *then* A *oscillates infinitely.*

If $A_n=\sum_1^n(-1)^{r-1}a_r$, then

$$A_{2n}=a_1-a_2+a_3-a_4+\ldots+a_{2n-1}-a_{2n}$$
$$=-\{(a_2-a_1)+(a_4-a_3)+\ldots+(a_{2n}-a_{2n-1})\}.$$

Since $a_{r+1}/a_r\to k>1$ as $r\to\infty$, there exists a number n_0 such that

$$a_{r+1}>k_0\,.\,a_r \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

for $r>n_0$, where $k>k_0>1$.

Hence A_{2n} is steadily decreasing for $2n>n_0$, since $a_r>0$.

Therefore $A_{2n}\to-\infty$ or $A_{2n}\to l$, where l is finite, as $n\to\infty$.

Again,

$$A_{2n+1}=a_1+(a_3-a_2)+(a_5-a_4)+\ldots+(a_{2n+1}-a_{2n}),$$

which is steadily increasing for $2n>n_0$.

Thus $A_{2n+1}\to+\infty$ or $A_{2n+1}\to l'$, as $n\to\infty$, where l' is finite and not in general equal to l.

Also

$$A_{2n+1}-A_{2n}=a_{2n+1}. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Considering (i), let c be the least value of r for which $a_{2r+1}>k_0\,.\,a_{2r}$. Then this inequality is also satisfied by all $r\geqslant c$, so that, writing

$$2n+1=2c+n',$$

we obtain

$$\begin{aligned}
a_{2c+1} &> k_0\,.\,a_{2c},\\
a_{2c+2} &> k_0^{\,2}\,.\,a_{2c},\\
a_{2c+3} &> k_0^{\,3}\,.\,a_{2c},\\
&\vdots\\
&\vdots\\
a_{2c+n'} &> k_0^{\,n'}\,.\,a_{2c}.
\end{aligned}$$

Also $n'\to\infty$ when $n\to\infty$.

Hence $a_{2n+1}=a_{2c+n'}>k_0^{\,n'}\,.\,a_{2c}\to\infty$ as $n\to\infty$, since

$$k_0>1. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

Then by (ii) and (iii) we obtain

$$A_{2n+1}-A_{2n}\to\infty \text{ as } n\to\infty.$$

Hence the statements $A_{2n}\to l$, $A_{2n+1}\to l'$, as $n\to\infty$, cannot both be true; therefore A oscillates infinitely.

Applications.

I. The series $1 - 2x + 3x^2 - 4x^3 + \dots$.

This may be shown to be convergent for $|x| < 1$ and by inspection is easily seen to oscillate infinitely for $x = 1$. To examine its behaviour for $|x| > 1$, we have with our previous notation that

$$\frac{a_{r+1}}{a_r} = \frac{r+1}{r} \cdot x \to x \text{ as } n \to \infty .$$

Thus if $x > 1$, the series oscillates infinitely, by the test. If $x < -1$, the series diverges: this is easily seen by putting $x = -y$ and applying a test. Hence the series converges for $|x| < 1$. It plainly diverges if $x \leqslant -1$. It also oscillates infinitely for $x \geqslant 1$.

II. The series $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$.

Here $$\frac{a_{r+1}}{a_r} = \frac{x^{2r+1}}{2r+1} \cdot \frac{2r-1}{x^{2r-1}} = \left(\frac{2r-1}{2r+1}\right) x^2 \to x^2 \text{ as } r \to \infty .$$

Thus the series oscillates infinitely for $|x| > 1$, by the test. It is easily shown to be convergent for $-1 < x \leqslant 1$ and divergent for $x = -1$.

Hence series converges for $-1 < x \leqslant 1$, diverges for $x = -1$, and oscillates infinitely for $|x| > 1$.

III. The series $\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \dots$.

Put $x = -y$. Then

$$\frac{a_{r+1}}{a_r} = \frac{y^{r+1}}{(r+1)(r+2)} \cdot \frac{r(r+1)}{y^r} = \frac{r}{r+2} \cdot y \to y \text{ as } r \to \infty .$$

Thus the series oscillates infinitely for $y > 1$, *i.e.* for $x < -1$. Clearly it converges for $|x| \leqslant 1$ and diverges for $x > 1$.

I. B. BUTTERWORTH.

2182. *Conditions for a developable surface.*

1. The conditions for a developable surface when the equation to the surface is given as $F(x, y, z) = 0$ or in the two-parameter form may be proved very simply by the use of a new form of definition of a developable surface. This new definition is given below in the form of a necessary and sufficient condition, the proof of which shows that the new definition is equivalent to any of the conventional definitions.

Two well-known theorems used in the proof are: (*a*) the tangent plane to a developable surface is the same at all points of a generator; (*b*) if the second curvature K of a surface vanishes identically, the surface is developable.

2. A necessary and sufficient condition that a surface be developable is that, for every point P on the surface, there is a neighbouring point P' on the surface such that the normals at P and P' are parallel.

The condition is necessary: for, if P' be any point on the generator through P, the tangent planes at P, P' are the same, and therefore the normals at P, P' are parallel.

The condition is sufficient. For, if $\mathbf{n}$ be the unit normal at P, and if $u =$ constant, $v =$ constant be the parametric curves, then there is a direction $du : dv$ through P for which $d\mathbf{n} = 0$, that is,

$$\mathbf{n}_1 du + \mathbf{n}_2 dv = 0.$$

But $$H^2\mathbf{n}_1 = (FM - GL)\,\mathbf{r}_1 + (FL - EM)\,\mathbf{r}_2,$$

and $$H^2\mathbf{n}_2 = (FN - GM)\,\mathbf{r}_1 + (FM - EN)\,\mathbf{r}_2.$$

Substituting and equating to zero the coefficients of $\mathbf{r}_1$ and $\mathbf{r}_2$,

$$(FM - GL)du + (FN - GM)dv = 0,$$
$$(FL - EM)du + (FM - EN)dv = 0.$$

The eliminant of these equations reduces to

$$(EG - F^2)(LN - M^2) = 0,$$

whence $LN - M^2 = 0$, since $EG - F^2 \neq 0$. Thus the second curvature K vanishes at every point, and the surface is developable.

3. A necessary and sufficient condition that the surface whose equation is $F(x, y, z) = 0$ be developable is that

$$\begin{vmatrix} F_{xx} & F_{xy} & F_{xz} & F_x \\ F_{yx} & F_{yy} & F_{yz} & F_y \\ F_{zx} & F_{zy} & F_{zz} & F_z \\ F_x & F_y & F_z & O \end{vmatrix} = 0.$$

For the direction ratios of the normal at a point $P(x, y, z)$ on the surface are F_x, F_y, F_z. At a neighbouring point P' they are $F_x + dF_x, F_y + dF_y, F_z + dF_z$. From section 2, the surface is developable if and only if there is a direction (dx, dy, dz) along the surface such that these direction ratios are proportional, that is, if

$$dF_x/F_x = dF_y/F_y = dF_z/F_z = d\lambda, \text{ say,}$$

that is, $$dF_x - F_x d\lambda = 0, \text{ etc.}$$

Expanding,
$$F_{xx}\,dx + F_{xy}\,dy + F_{xz}\,dz - F_x\,d\lambda = 0,$$
$$F_{yx}\,dx + F_{yy}\,dy + F_{yz}\,dz - F_y\,d\lambda = 0,$$
$$F_{zx}\,dx + F_{zy}\,dy + F_{zz}\,dz - F_z\,d\lambda = 0,$$

Also,
$$dF \equiv F_x\,dx + F_y\,dy + F_z\,dz = 0,$$

since (dx, dy, dz) is a displacement along the surface. Eliminating $dx, dy, dz, d\lambda$ between these four equations, the required result is obtained. The condition is therefore necessary, and its sufficiency is easily proved by reversing the steps of the above argument.

4. A necessary and sufficient condition that the parametric equations $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ represent a developable surface is that

$$\begin{vmatrix} l_u & l_v & l \\ m_u & m_v & m \\ n_u & n_v & n \end{vmatrix} = 0$$

where $l = y_u z_v - z_u y_v$, $m = z_u x_v - x_u z_v$, $n = x_u y_v - y_u x_v$.

For, the direction ratios of the normal at P are (l, m, n) as given above. Reasoning as in section 3, there is a displacement (du, dv) such that

$$dl/l = dm/m = dn/n = d\lambda, \text{ say,}$$

that is, $dl - l d\lambda = 0$, etc. Expanding,

$$l_u du + l_v dv - l d\lambda = 0,$$
$$m_u du + m_v dv - m d\lambda = 0,$$
$$n_u du + n_v dv - n d\lambda = 0.$$

Eliminating $du, dv, d\lambda$ between these three equations, the result is obtained.

5. It is well known that the condition that the equation $z = f(x, y)$ should represent a developable surface is that $f_{xx} \cdot f_{yy} = (f_{xy})^2$. This may be proved independently by writing $f_x, f_y, -1$ for the direction ratios of the normal and

proceeding as in sections 3 and 4, or it may be deduced from the condition of section 3 by writing $F(x, y, z) = f(x, y) - z$, or from the condition of section 4 by writing $x = u$, $y = v$, $z = f(u, v)$.

6. A parabolic point on a surface may be defined as a point at which the second curvature K is zero. If the conditions of any one of sections 2, 3, 4, 5 are satisfied for a single point P on the surface, then $K = 0$ at P, and P is a parabolic point. Conversely, if P is a parabolic point, the conditions will be satisfied at P. The proofs of these statements are the same as the proofs given above for a developable surface, except for the proof of necessity in section 2. This may be proved by using Rodrigues' formula for the displacement in a principal direction :

$$\kappa d\mathbf{r} + d\mathbf{n} = 0,$$

where κ is the corresponding principal curvature. For if $K = 0$ at P, then one of the principal curvatures κ vanishes at P, so that for a displacement in the corresponding principal direction, Rodrigues' formula shows that $d\hat{\mathbf{n}} = 0$. Therefore the normal at a neighbouring point P' in this direction is parallel to the normal at P.

7. The condition of section 3 may be used to give a simple proof of the following theorem :

If a central conicoid contain a parabolic point, the conicoid is developable and is either a cylinder or a cone.

For the equation to any central conicoid may be written

$$ax^2 + by^2 + cz^2 - d = 0, \qquad \text{.....(1)}$$

by a suitable choice of axes. The condition for a parabolic point is

$$\begin{vmatrix} 2a & 0 & 0 & 2ax \\ 0 & 2b & 0 & 2by \\ 0 & 0 & 2c & 2cz \\ 2ax & 2by & 2cz & 0 \end{vmatrix} = 0$$

which reduces to $$abc(ax^2 + by^2 + cz^2) = 0.$$

By virtue of (1) this may be written $abcd = 0$. If this is satisfied at one point on the conicoid, it is satisfied at every point, since the left-hand member is constant. Thus $K = 0$ at all points of the conicoid, and it is developable.

Also, if $abcd = 0$, at least one of the constants a, b, c, d vanishes. If $d = 0$, the conicoid is a cone. If one of a, b, c vanishes and $d \neq 0$, it is a cylinder. If two or three of the constants vanish, it is a pair of planes, which may be regarded as a special case of either a cone or a cylinder.

Similarly, if a paraboloid contain a parabolic point, it is developable and is a cylinder. For the equation to any paraboloid may be written

$$ax^2 + by^2 + z = 0,$$

the condition for a parabolic point reduces to $ab = 0$, and the proof proceeds as before. E. J. BURR.

2183. *On sub-factorial n.*

H. E. Piggott (*M.G.*, Vol. XXXIII, 1949, pp. 214, 215, Note 2063) and R. L. Goodstein (*M.G.*, Vol. XXIII, 1939, pp. 205, 206, Note 1363) have shown that, if u_n is the number of ways in which a row of n elements may be so deranged that no element shall be in its proper place, then

$$u_n = n!\left\{\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} \ldots (-1)^n \frac{1}{n!}\right\}, \qquad \text{.....(1)}$$

i.e. the integer nearest to $n!/e$. The former derived this expression from the recurrence relation

$$u_n = (n-1)(u_{n-1} + u_{n-2}). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

It is of interest to note that n ! satisfies this relation, and we can use this fact to show that the expression (1) is the second fundamental solution of (2).

It is well known that, if y_1 is one solution of the linear homogeneous differential equation of the second order $y'' + py' + qy = 0$, the second fundamental solution is

$$y_2 = y_1 \int_0^x \left(\exp - \int_0^x p\, dx\right) \Big/ {y_1}^2 . dx.$$

The corresponding property of the linear homogeneous recurrence relation of the second order $u_{n+2} + pu_{n+1} + qu_n = 0$ is that, if U_n is one solution, the second fundamental solution is

$$V_n = U_n \sum_1^{n-1} \left\{ \left(\prod_1^{n-1} q_n \right) \Big/ U_n U_{n+1} \right\}.$$

Applying this to (2) with $U_n = n!$ and $q_n = -(n+1)$, the expression (1) is immediately derived.

Whitworth (*Choice and Chance*, 4th edition, 1886, p. 107) gave the name *sub-factorial* n to the function defined by (1) and denoted it by $\lfloor\!\lfloor n$ (corresponding to the notation $\lfloor n$ for factorial n). Chrystal (*Algebra*, Part II, 2nd edition, 1919, p. 25) suggested $n_!$ as a more convenient notation, to correspond to $n!$. Whitworth chose the term *sub-factorial* because the method of building up the function is very similar to that for factorial n. The mode of formation is: begin with unity and multiply successively by 1, 2, 3, ... as if we were going to form n !, but let every even product be increased by unity and every odd product be diminished by unity:

$$\begin{aligned}
1_! &= 1 - 1 = 1\,! - 1 && = 1!(1 - 1/1!),\\
2_! &= 2(1-1) + 1 = 2\,! - 2 + 1 = 2!(1 - 1/1! + 1/2!),\\
3_! &= 3(2(1-1)+1) - 1 && = 3! - 3\,.\,2 + 3\,.\,1 - 1\\
&&& = 3!(1 - 1/1\,! + 1/2! - 1/3!),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
n_! &= n(\ldots(4(3(2(1-1)+1) - 1) + 1) \ldots (-1)^n) + (-1)^n\\
&= \text{the expression (1)}\\
&= n!\, e^{-1} + H, \quad n \text{ even},\\
&= n!\, e^{-1} - H, \quad n \text{ odd},
\end{aligned}$$

where $$1/(n+1) > H > 1/(n+2),$$

so that $n_!$ is the integer nearest to $n!\, e^{-1}$.

Derivations of the expression (1) and related propositions, with applications to problems in probability, are given by Whitworth (*loc. cit.*, pp. 107-114), Chrystal (*loc. cit.*, pp. 22-25), Levy and Roth (*Elements of Probability*, 1936, p. 37 ff.), Uspensky (*Introduction to Mathematical Probability*, 1937, p. 108 ff.), Barnard and Child (*Higher Algebra*, 1936, pp. 498, 499). S. RUSHTON.

2184. *On the operation of derivation.*

In a note bearing the above title, which is to appear in the *Rendiconti dei Lincei*, I present a new definition of the derivative which modifies the basic notion.

Let $f(x)$ be a real function of the real variable x defined in a set I_x of points

$x>0$ admitting $x=0+$ as a point of accumulation (which may or may not belong to I_x). We then define the *oscillatory derivative* of $f(x)$ at $x=0+$ as

$$\lim\,(\Omega_{0,x}/x)=\lim\,\{(L_{0,x}-l_{0,x})/x\}$$

as $x\to 0+$ in I_x, where $L_{0,x}$ and $l_{0,x}$ are the upper and lower limits of $f(x)$ in I_x. This limit may be finite or infinite.

We then have :

Theorem I. If $y=f(x)$ is defined in I_x (where $x>0$) with x as point of accumulation, and if it is defined also at $x=0$, then, if the oscillatory derivative at $x=0+$ exists and is zero, the ordinary derivative exists, for x varying in I_x, at $x=0+$, and is likewise zero.

Theorem II. If $y=f(x)$ is such that

(i) it is continuous in $0\leqslant x\leqslant a$;

(ii) the ordinary right-hand derivative exists at $x=0$;

(iii) for each x in the above interval there exists an $x_1>0$ at which $f(x)$ attains its maximum $M_{0,x}$, and there exists an $x_2>0$ at which $f(x)$ attains its minimum $m_{0,x}$ in the interval ; then the right-hand oscillatory derivative of $f(x)$ exists at $x=0$ and is zero.

Thus if a continuous function possesses an ordinary right-hand derivative at a point, it possesses an oscillatory right-hand derivative there which coincides (sometimes only in absolute value) with the former. On the other hand, the existence of the oscillatory derivative implies the existence of the ordinary derivative if, and only if, the oscillatory derivative is zero.

S. MINETTI (Rome).

2185. *Riccati's differential equation.*

The purpose of this note is to apply, to the above equation, the method for solving a Bernoulli linear equation given by Professor Neville in the *Gazette* (XVIII, 1934, p. 321), where he remarks that the equation

$$y'+Py=Qy^n$$

can be multiplied by an integrating factor I to give

$$(yI)'=(QI^{n-1})\,.\,(yI)^n.$$

1. Any Riccati equation of the form $y'=P+Qy+Ry^2$ can be reduced to $z'=P_1+R_1z^2$. For, on multiplying the given equation by an integrating factor $I[=\exp(-\int Q\,dx)]$ we have

$$(yI)'=PI+(R/I)\,.\,(yI)^2,$$

which is of the required form. The usual results relating the solutions of a Riccati equation can be obtained with rather less computation using the latter form of equation.

2. When one integral $z=r(x)$ is known it is usual to transform the equation by the substitution

$$z=r(x)+1/u(x).$$

This substitution appears artificial, but the more natural one

$$z=r(x)+u(x)$$

can be made without lengthening the work if the resulting Bernoulli is solved by the method mentioned.

A. BUCKLEY.

2186. *Sylvester's law of nullity.*

Suppose that $A \,.\, B = C$, where A, B and C are matrices of dimensions $m \times n$, $n \times p$, $m \times p$, and rank r, s, t respectively. We shall give short proofs of the following well-known results :

$$t \leqslant r, \; t \leqslant s \; ; \quad t \geqslant r + s - n \text{ (Sylvester's law of nullity).}$$

1. If $Bx = 0$, then $Cx = 0$, where x is a p-column vector, so the solutions of $Bx = 0$ are a subset of the solutions of $Cx = 0$, But the solutions of $Cx = 0$ form a linear space S_{p-t} of dimension $p - t$ and the solutions of $Bx = 0$ form an S_{p-r}. Hence $p - r \leqslant p - t$ and so $t \leqslant r$. Transposing the matrices it follows that $t \leqslant s$.

2. We can get all solutions of $Cx = 0$ by solving first the equation $Ay = 0$ and then the equation $Bx = y$ when compatible. The set of solutions of $Ay = 0$ form an S_{n-r}, and those solutions for which the equation $Bx = y$ is solvable form a subset of this set. The solutions of $Cx = 0$ form an S_{p-t}. Hence $n - r \geqslant p - t$, so that $t \geqslant p + r - n$. By the previous result it follows that

$$t \geqslant r + s - n. \qquad \text{G. A. Dirac.}$$

2187. *On Note* 2070 *: the equation* $U_1^2 S - 2U_1 U S_1 + U^2 S_{11} = 0$.

Dr. Maxwell has pointed out that if $S = 0$ be a quadric and P_1 a given point, then the above equation has two interpretations :

(1) If $U = 0$ is a plane, the equation determines the cone joining P_1 to the conic in which U and S intersect.

(2) If $U = 0$ is a quadric, it determines the quartic surface which is the locus of a point P such that the points P and P_2 are conjugate with respect to U and collinear with P_1, and P_2 lies on S.

Is not the connection due to the double use of the symbols U, U_1 rather than to a true mathematical relationship?

For case (1) can be considered as the locus of a point P such that P and P_2 are collinear with P_1, while P_2 lies on the conic determined by U and S. Hence, since in (1) P_2 lies on $U = 0$ and in (2) P_2 and P are conjugate with respect to $U = 0$, we have, in both cases, with the usual notation,

$$U_2 = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Also in both cases P_2 lies on $S = 0$, and hence

$$S_{22} = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

and P, P_1, P_2 are collinear, so that

$$P_2 = k_1 P_1 + kP. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

Substituting from (iii) in (i) and (ii), we have

$$k_1 U_1 + kU = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}$$

and

$$k_1^2 S_{11} + 2k_1 k S_1 + k^2 S = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(v)}$$

and so eliminating the ratio $k_1 : k$ from these equations we obtain the locus of P in both cases as

$$U^2 S_{11} - 2U_1 U S_1 + U_1^2 S = 0.$$

It is seen, therefore, that the method is identical, the one difference being the double meaning attached to the condition (i) according as $U = 0$ represents a plane or a quadric.

In Note 2070, case (2) was misquoted, the S and U being interchanged.

G. Allman.

[Dr. J. L. B. Cooper and Mr. A. H. Wallace have also supplied answers to Dr. Maxwell's query.—Ed.]

REVIEWS.

Leçons sur la Théorie des Fonctions (Principes de la théorie des ensembles en vue des applications à la théorie des fonctions). Par ÉMILE BOREL. 4th edition. Pp. xiii, 295. 1,200 fr. 1950.

Leçons sur les Nombres Transfinis. Par W. SIERPINSKI. (Rep.) Pp. vi, 240. 1950. Monographies sur la théorie des fonctions. (Gauthier-Villars)

The fourth edition of No. 1 in the Borel series is a welcome reprint; the material added to the third edition consists of a dozen or so footnotes underlining the importance which Borel now attaches to calculability and effective definition in the theory of sets. There is also a new note on the axiom of choice and asymptotic definitions, that is, definitions in which number sequences, for example the digits of an infinite decimal, are considered as equivalent if they differ only in a finite number of their terms. From the empiricist's point of view, such a classification is suspect; the agreement in the first million places of the decimal expansions of

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots \quad \text{and} \quad \tfrac{1}{2}\int_0^1 \frac{dx}{\sqrt{(1-x^2)}}$$

tells us exactly nothing about the equivalence of these numbers. The substance of the note is to show that the selection axiom, when applied to such a classification, leads to paradoxes similar to that of Hausdorff (the division of the surface of a sphere into disjoint sets A, B, C, D, such that D is countable, and A can be transformed into B and into C, and also into $B+C$ by suitable rotations). However, the author of the paradox that the set of all rational numbers can be covered by a sequence of intervals of arbitrarily small total length, can hardly complain if a generation has grown up which is more attached to Zermelo than to Euclid.

The 1898 edition of Borel's monograph is still a remarkable document and a tribute to the many-sidedness of Borel's great contribution to mathematics. In 1898 he was a pioneer, applying Mengenlehre to the theory of functions, real and complex, taking the first steps in the theory of measure, and applying set theory to probability. The reaction of French mathematicians to Cantor's theory is most interesting; the sceptical admiration of Poincaré gave way to the restrained enthusiasm of Borel; Lebesgue was less exacting than Borel, and finally, in Bourbaki, we come to the axiomatic approach in mathematics which *divides* the difficulties so as to overcome them separately. Borel has never consented to these tactics. His theory of measure is an illustration: Denjoy has pointed out that Borel's measure theory, properly understood, is just as extensive as Lebesgue's (since Borel sets and null sets will build all measurable sets); but it is Lebesgue's theory that has made the greater impact on analysis. With Borel there is always a preoccupation with the calculable, the definable, the constructible, which Lebesgue has been content either to ignore or to interpret less exactingly than Borel. What gives Borel's book its great piquancy, apart from the mathematical achievements which are now classical, is his exposition of the empiricists' position, which as far as he is concerned is to the left of Poincaré but to the right of Brouwer. His book contains the famous exchange of letters between Hadamard, Baire, Lebesgue and Borel regarding the validity of Zermelo's theorem; it also has an illuminating discussion between R. Wavre and P. Lévy on Brouwer's logic. In the borderland between mathematics and philosophy there are many irate and dogmatic guides, and for the student who is on the threshold of mathematics there is perhaps no better source at which to start than the genial writings of Borel. In an article written in 1935, Borel expressed himself

as follows : " Si l'on me demandait de caractériser par un trait commun la méthode que j'ai appliquée . . . je crois que c'est un souci constant d'étudier les êtres mathématiques en eux-mêmes, comme le biologiste étudie les êtres vivants . . . cette méthode de travail m'a conduit à une conception réaliste des mathématiques, qui distingue les êtres mathématiques pouvant être effectivement définis de ceux dont l'existence est purement hypothétique, mais il n'est point ici le lieu de reprendre une discussion sur les définitions où intervient l'infini, discussion qui ne sera sans doute jamais close."

It was to further this discussion that Borel was glad to incorporate Sierpiński's *Nombres Transfinis* in his Collection, and to recommend it, when it first appeared in 1928, as a brilliant exposition of a point of view diametrically opposed to his own. The new edition is a reprint of the old ; it remains the most readable account of Cantor's great work : it is self-contained, and no specialised mathematical knowledge is needed to appreciate it. It is a study of the transfinite for its own sake and with a view to applications that are wider than Borel has in mind. The Polish school has led the world in exploiting just those parts of set theory which the empiricists mistrust, and it is typical that where Borel is troubled at the prospect of a bounded set being geometrically equivalent to a proper subset of itself, Sierpiński produces an endless supply of rabbits from the Zermelo hat, and he recently proved (*Fundamenta Math.*, 1945) that every three-dimensional sphere S is decomposable into $2^{\aleph_0}$ disjoint sets, each of which may be broken up into nine parts which serve to build a sphere identical with S !

No mathematical library should be without these books, and no student of mathematics can fail to enjoy them.

H. Kestelman.

La géométrie contemporaine. Par A. Delachet. Pp. 128. 1950. Collection " Que sais-je ? ", 401. (Presses Universitaires de France, Paris)

This is a bright little book which outlines (i) the notion of a group, (ii) abstract geometries and spaces, and (iii) topology.

There are illustrative examples, and the presentation is skilful. This can also be said of the frequent compliments paid to contemporary mathematicians. For example, on p. 53 we find this : " Le passage des surfaces algébriques aux hypersurfaces algébriques, c'est-à-dire aux variétés algébriques à plus de deux dimensions plongées dans un espace à plus de trois dimensions provoque des difficultés d'essence différente. C'est pourtant dans cette voie que les géomètres contemporains et en particulier M. Godeaux dirigent leurs pas." We can only hope that this sentence will not form the theme of an Italian counterblast at the next International Congress of Mathematicians. In any case, such partiality should not prevent this book, which is pleasing, from being read by all amateurs of geometry.

D. Pedoe.

Reine Mathematik. I, II. Edited by W. Süss. Pp. 316, 265. D.M. 10 each. 1948. Naturforschung und Medizin in Deutschland, 1939–46, 1, 2. (Dieterich'sche Verlagsbuchhandlung, Wiesbaden)

The painstaking thoroughness of German scholarship is proverbial. In two solid and well-printed volumes we have a summary of the contributions made by German mathematicians to our subject during the years 1939-1946. Although, of course, such a summary gives a picture which is bound to be incomplete and even one-sided, it shows clearly the main lines of German work, and since the references are full and detailed the volume should assist those who during the years in question had no access to German periodicals, access which is not even now as easy as it was before 1939.

It is impossible to comment on all the sections; to pick out one or two would reflect only an individual preference, and it will probably be more useful to give readers a list of the sections and of the authors responsible for them. History (Hofmann); Foundations (Lorenzen); Elementary mathematics (Zacharias); Algebra and theory of numbers (Hasse); Group-theory (Zassenhaus); Lattice-theory (Köthe); Theory of sets and of real functions (Nöbeling); Infinite sequences and methods of summation (Knopp); Almost-periodic functions (Maak); Special functions of mathematical physics (Magnus); Series of expansions in mathematical physics (Lense); Theory of functions (Kneser and Ullrich); Modular and automorphic functions (Petersson); Ordinary differential equations (Müller); Partial differential equations of the first order and Pfaff's problem (Bilharz); Potential theory (Maruhn); Partial differential equations of order greater than 1 (Pinl); Special boundary problems (Buchholz); Calculus of variations (Boerner); Integral equations (Tautz); Eigen-value theory (Wielandt); Functional analysis and integral transforms (Köthe); Foundations of geometry (Sperner); Analytic geometry (Süss); Algebraic geometry (Deuring); Differential geometry (Bol); Projective relativity theory and Cosmology (Jordan); Convexity theorems and differential geometry in the large (Gericke); Integral geometry (Maak); Topology (Seifert and Threlfall). T. A. A. B.

Calcul opérationnel. Par E. LABIN. Pp. 150. 780 fr. 1949. (Masson, Paris)

M. Labin's own words make out a strong case for adding still another book to the many volumes which have been written about operational calculus in the last thirty years: " Dans l'étude d'un problème pratique, un travailleur — physicien, chimiste, ingénieur, naturaliste, économiste, mathématicien même — tombe sur une technique mathématique dont il aperçoit la possible fertilité mais dont il n'a pas le maniement présent à l'esprit. Il ne peut pas et ne veut pas consacrer, à maîtriser cette technique, le temps et les efforts mentaux qu'il a consacrés, lorsqu'il était étudiant, à apprendre l'ensemble de l'analyse. Ce qu'il désirerait, ce serait seulement pouvoir réviser les résultats utiles, les résultats opératoires de cette technique, assez rapidement mais assez efficacement pour se mettre à même de l'appliquer à son problème."

But how is this to be done? The crude reply which asserts that formulae and tricks of technique are enough, that no theory is needed, is rejected, quite properly, by M. Labin. He wishes, indeed, to write a book of results, but a skeleton of mere formulae would, he knows, be unsatisfactory just as the " Laplace transform made easy " type of book is unsatisfactory. What is said must be the truth, if not the whole of the truth, and it must be set out in logical order. But the logical order demanded is not that of the mathematician, but of the user. Results must be arranged in the order most convenient for the man who is going to use the Laplace transform as a tool, not in that most convenient for exposition of the theory. Complete generality is not attempted; and theorems must be enunciated, not in the formal manner of the *Cours d'Analyse*, but as answers to the questions the tool-user is most likely to ask. All this M. Labin explains in his preface, in five pages well worth reading.

The actual contents of the book, then, cannot be particularly novel; but they are arranged in an unusual order, with constant appeals to the eye through tables and diagrams, as well as through the emphasis of Clarendon and italic founts. The reader is expected to have some knowledge of analysis: the economist, for example, who is shaky on uniform convergence, will have cause to cry that this is not " Heaviside without tears ". In all, the value of M. Labin's book lies in the quick reference it gives to the points we had for-

gotten, or had happened never to learn. For those to whom the Laplace transform is an entirely new technique, a wise course would be to read some brief general account (such as that of Curry in the *American Mathematical Monthly*, 50) and then turn to M. Labin for more information on points of special interest. T. A. A. B.

Numerical Calculus. By W. E. Milne. Pp. vi + 393. 21s. 1949. (Princeton University Press ; London, Geoffrey Cumberlege)

The title of this book may call to mind Whittaker and Robinson's well-known text ; the scope and method are, however, somewhat different. For one thing, the emphasis here is less on theory than on technique ; and this, in spite of the fact that the formulae and many of the devices, for example, the use of orthogonal functions, have an elegant theoretical background. But the more important point is that the book takes account of the reorientation towards numerical calculation brought about by the desk calculating machine. In the pre-machine epoch the labour of heavy multiplication and division was not precisely shirked, but at any rate circumvented as far as possible by logarithms and other devices of a pen-and-paper nature. The situation is now changed ; the formation and accumulation of binary products, with subsequent division, is precisely the operation in which the machine revels ; and numerical calculus has acquired in consequence a variety, amounting at times to an *embarras de choix*, of new methods, or a revival of old methods formerly regarded as impossibly laborious.

The author, well aware of these matters from his own experience, has successfully composed his book in the light of them. He has managed to include all those topics that are *de rigueur* : solution of simultaneous linear equations, of algebraic (and certain transcendental) equations by successive approximation, interpolation, numerical differentiation and integration and solution of differential equations, a summary of finite differences and difference equations, the fitting of polynomials and of harmonic terms by least squares. Interpolation by rational fractions, involving " reciprocal differences ", is also included ; and there are some thirty pages of useful appendices and tables.

The treatment is straightforward and fresh. The preface frankly says : " the presentation is intentionally elementary . . . mathematical elegance and rigour have frequently been sacrificed in favour of a purely naive treatment ". This is true enough : none the less the book, photo-set from carefully typed copy and easy to read, would serve as a very useful text for a course in a mathematical laboratory supplied with a few desk machines and good modern tables. A. C. A.

An introduction to topology. By Solomon Lefschetz. Pp. viii, 218. 32s. 1949. (Princeton University Press ; Geoffrey Cumberlege, London)

As the title suggests, this latest volume in the Princeton Mathematical Series differs from many of its predecessors in that it is a textbook rather than a monograph. It is not intended as a first introduction to topology, but is aimed rather at the graduate student who is just beginning research in topology. Accordingly, it presupposes a knowledge of elementary point-set theory, and seeks to expound the basic methods and results of algebraic topology. The student who meets these ideas here for the first time should find the account given at once formally adequate and highly stimulating, for in this exposition the detailed algebraic structure and the underlying geometrical ideas are not made to compete for the attention, but complement one another, as they should. And, indeed, the book can be recommended to any mathematician who wants to know what algebraic topology is about.

Professor Lefschetz begins with an informal account of some of the questions topology is intended to answer, and of some of its contacts with the rest of mathematics. Then, after a chapter summarising the point-set topology to be assumed, the main business of the book begins with a chapter on polyhedra of dimension at most two. Apart from the fact that only chains modulo two are considered, most of the later ideas are introduced here already, sometimes in a simple form, *e.g.* complexes and their subdivisions, chain-mappings, and the " interlocking systems " which relate the algebraic constructions to the underlying geometry. The chapter contains a proof of the Jordan curve theorem, and an account of the normal form theory of closed surfaces.

The remainder of the book is devoted to polyhedral topology in n dimensions. A chapter on the homology theory of complexes is followed by one on their transformation theory, including, for instance, the Brouwer fixed point theorem and Hopf's theorem on the mappings of n-spheres on n-spheres. The theory of homotopy is further developed in a chapter which includes a proof of the topological invariance of the homology groups that is inspired by the Čech homology theory, and some account of the fundamental group and of the higher homotopy groups of Hurewicz. The book ends with a chapter on manifolds, covering relative homology theory, and the Poincaré and Alexander duality theorems.

The printing and general lay-out of the book are up to the best standards of a series that has been somewhat uneven in this respect ; and the many figures have been well printed. There is a certain carelessness over matters of detail in the book's production that will annoy the precise ; the town of Königsberg, for instance, receives one spelling in the text, and another in the index (under *problem*), and neither is the usual modern spelling. However, this carelessness certainly does not extend to the author's choice of material, or to the presentation of his arguments ; and it is only a superficial blemish on a very good book indeed. G. H.

Plane Trigonometry. By H. M. DADOURIAN. Pp. 109, 84. $3. 1950. (Addison-Wesley Press, U.S.A.)

This relatively short Trigonometry book raises a vitally important question with regard to the function of a textbook. The nature of the question will be apparent when it is realised that the text begins by defining the " trigonometric " ratios of an angle of any magnitude in terms of x, y, r and θ, the Cartesian and polar coordinates of a point, and proceeds in a later section to consider the ratios of the positive acute angle and their relationship with the sides of the " right " triangle as a special case of the general result. This form of synthesis may possibly be of value to mature American students at, or near, the University level, but it would clearly be quite useless for immature English schoolboys and schoolgirls who commence the study of trigonometry long before they ever hear of polar coordinates. Such a method of presentation seems to resemble swimming upstream, and it is surely preferable (though perhaps less logical) to follow the historical course of the development, where the easy form is treated first and, later, generalisations are derived from it. Is it, in fact, the function of a textbook to present a final logical synthesis of the subject-matter, or is its function to arrange the subject-matter in such a way as to facilitate the progress of the average student meeting new ideas for the first time?

Aside from this quite crucial issue, the book contains some useful material. There is a good chapter on vectors and the application of trigonometry to statics, the section on the relationship between the ratios of any angle and those of the " reference angle " is concise and well-arranged, and the usual

work on graphs, identities, equations and the solution of " oblique " triangles is properly covered. Proofs of the addition formulae are given only for the case where $(\alpha + \beta)$ is acute, with a somewhat naive note to the effect that " it can be proved . . . "

It is perhaps ungenerous to complain that a book entitled *Plane Trigonometry* contains no three-dimensional work. The complaint is not quite so illogical as it might appear, since the use of trigonometry in the fundamental process of analysing three-dimensional figures into plane triangles is so illuminating that no student should be allowed to miss it.

Three minor points merit a passing mention. In dealing with bearings, the three-figure method is not explicitly described, though it is indeed hinted at. The Greek word for triangle is not " trigon " as stated in the text. And cannot something be done to lessen the optical illusion in the figure on p. 42?

The completeness of the five-figure tables supplied will be appreciated when it is noted that they occupy no fewer than 81 pages. G. L. P.

Examples in Engineering Mathematics, Bk. III. By W. DONALDSON. Pp. 96. 5s. 1949. (Harrap, London)

This book of examples for students taking third year courses for the National Certificates in Mechanical, Electrical and Aeronautical Engineering is the completion of a series ; Book I was reviewed in the *Gazette*, Vol. XXVII, page 187 ; and Book II in Volume XXXI, page 317.

The object of the book is to provide sufficient examples for classwork, homework and revision following the scheme of the first and second year books of the series.

While there is no actual text as such, here and there hints are given to set the student on the method of work. There are 28 sets of examples with four test-papers at sectional intervals. The first five sets are algebraical, including various types of equation, graphical solutions and the binomial theorem. Then follows examples on differential calculus, functions, limits, problems involving trigonometrical functions, products, quotients, velocities and accelerations, maxima and minima, logarithmic and exponential functions and finally integration, S.H.M., mean values, partial fractions, areas and volumes, centres of gravity and moments of inertia. Thus it is seen that a full course is well covered. The examples are well selected and vary in number from twelve to twenty in a set.

The answers fill 10 pages, and trigonometrical tables, truly very useful, take another 10 of the total of 96 pages.

The printing of the indices is in many places somewhat unusual, particularly when letters are used, and in these cases would certainly cause concern to many, not yet proficient mathematicians, who would be using the book : the more mature could probably appreciate what was intended. E. J. A.

Exercises in Elementary Mathematics, Book I. By K. B. SWAINE. Pp. 186. 5s. 6d. 1949. (Harrap, London)

This is the first of a series of books of exercises planned to meet the needs of those who are working to the Alternate Syllabus in Elementary Mathematics of the School Certificate or what will be in future known as the Ordinary Stage of the Certificate of Education. The scheme aims to weld together the parts of Arithmetic, Algebra, Geometry and Trigonometry that belong to the Ordinary Stage by planning the order of work as a continuous course rather than a succession of separate sections. The scope of the work is wide to allow the teacher latitude to follow out his own ideas. There is hardly any book-work for, as the author says, " The ability to apply knowledge gained only comes with practice."

The examples throughout are good and have an air of reality which gives distinct interest.

The book opens with revision exercises on the four rules, money and compound quantities, averages and fractional parts. Simple geometrical ideas, starting with the brick and the match-box as examples of cuboids, serve as an introduction to cubes, cones and pyramids. Practical exercises in making these solids provide the material for the consideration of surfaces and areas, which lead directly to generalised arithmetic and the beginnings of algebra. Arithmetical factors give index notation to numbers, and the application to letters follows. Fractions in arithmetic and algebra are taken together and with them equations.

The ideas of direction is developed into the consideration of angles and the Mariner's Compass.

Here unitary method is introduced, both direct and inverse. The final question in this exercise is : " If a boy is 5 ft. tall when he is 12 years old, how tall will he be when he is 30 years old? " The answer rightly says, " No answer." Doubtless the question is introduced with the idea of making the boy think about the question before answering it. Unfortunately, there are many people, parents and others, who give the impression to children that examiners set questions to " catch out " the candidates, and this undoubtedly does cause an unnecessary amount of nervousness at examinations. Candidates look for what is not in the question.

It is an instruction to examiners by some examining bodies, and a generally-accepted principle by all examiners, that no question should be set which is in any way the nature of a trick, and one would suggest that no such question should appear in an ordinary textbook.

Decimal fractions, the metric system, approximations and decimalisations next find place as a prelude to the drawing and measurement of lines and angles, scale-drawing and plans. Here added interest is given to the harder examples by the inclusion of costs of building and furnishing.

Statistical graphs are in good variety. Some good work on parallels, triangles, the sum of the angles of triangles and polygons and more scale-drawing leads to a very clear exposition of directed numbers, first by a picture showing a bush, a tree, a pillar-box, a telegraph-pole, a house and a lamp-post, all in one line, and then by the well-known " stair-case " illustration to assist the fuller understanding of the concept, often difficult to the young mind.

The Revision Exercises, twenty-four in number for the whole year's course, are collected at the end of the book, but they can be used as additional exercises, as indicated, as the course proceeds.

The whole is well and clearly printed. This book should interest all teachers of the early stages of mathematics, whether in the grammar, modern junior technical school, for no pupil who has worked through this course, designed for the first year of the Certificate of Education Course, can fail to have anything but a very useful and desirable foundation upon which to build, whatever be the later developments and requirements by the pupil. E. J. A.

The Mathematical Theory of Communication. By CLAUDE E. SHANNON and WARREN WEAVER. Pp. 117. $2.50. 1949. (University of Illinois Press, Urbana)

Occasionally one reads a paper or a book of a mathematical nature which one finds difficult to fit into any orthodox classification of our science. Wiener's *Cybernetics* is one example, and the present booklet, dealing with a similar subject, is another. Connected with this difficulty of attaching a label to a study which deals with such subjects as thermodynamics, statistics and engineering or perhaps with none of them, there is the uncertainty about

where it all leads to. No obvious continuation is suggested, yet it is no surprise to learn that C. E. Shannon has gone on to deal with secrecy systems and to design—on paper—a machine which ought to play chess rather competently.

In any case, this is a very entertaining book. It is a reprint of an article by Shannon in the *Bell System Technical Journal*, followed by a paper by W. Weaver, which " may well be read first by those desiring a panoramic view of the field " (quoted from the Preface). The concept " communication " is used to include " all of the procedures by which one mind may affect another " (p. 95), but it is not specifically concerned with the meaning of what is being transmitted. The basic model assumes a " source of information " and a " channel " ; the former will " choose successive symbols according to certain probabilities depending, in general, on preceding choices as well as the particular symbols in question " (p. 10)—statisticians will be reminded of the generation of a Markoff chain—and the latter is " a system whereby a sequence of choices from a finite set of elementary symbols can be transmitted from one point to another " (p. 7). These descriptions must slightly be altered for continuous information. Noise is, of course, any perturbation of the signal during transmission. The theory becomes mathematical by the introduction of such quantitative concepts as the information of a source, which is the measure of the choice which one has in selecting a particular message, and the channel capacity which measures the average information transmitted during the unit of time. The mathematical form of the information is that of entropy in thermodynamics. The fundamental theorem states at what average rate symbols can be transmitted and how this rate can be maximised by skilled coding, which is conveniently done in binary form.

A number of side-lines are touched upon, which upset the balance of the treatment but make certainly for stimulating reading. To give an example, we have the notion of redundancy, which indicates how much of the transmission is determined by the structure of the source. The English language appears to have 50 per cent redundancy, *i.e.* half of what we write (in non-Shavian spelling) is imposed by the structure of our language. If the probability of a letter in a word were independent of the preceding one, then it would be trivial to construct crossword puzzles, since any two-dimensional array would do. If the redundancy were only 33 per cent, then it would be possible to construct three-dimensional crosswords (but the author does not say whether this would be a blessing).

The mathematician will find the treatment of the discrete sequence of symbols fairly satisfactory. However, when it comes to continuous transmission, he has to accept the assertion that " the occasional liberties taken with limiting processes can be justified in all cases of practical interest " (p. 49).

Weaver's exposition stresses that the accuracy of the transmission of symbols has some bearing on the conveyance of meaning, but the reviewer found all this rather vague, and he is uneasy at the suggestion that " Eddington would have been willing to include *meaning* along with beauty and melody " (p. 117, the reference is to a passage in *The Nature of the Physical World*). At this early stage a mathematical theory had perhaps better be based on a calculus rather than on a mental extrapolation, however ingenious.

S. V.

Photoelasticity : Principles and Methods. By H. T. Jessop and F. C. Harris. Pp. viii + 184. 28s. 1949. (Cleaver-Hume Press)

There has in recent years been a considerable revival of interest in photo-

elasticity as a practical method of experimental stress analysis. This book is therefore very welcome in giving a comprehensive and up-to-date review of the present state of knowledge of this technique, together with much useful practical advice based on the authors' experience in their laboratory at University College, London.

The first four chapters of this book summarise very lucidly the portions of the theory of elasticity and of optics which the reader must understand before approaching Chapters V and VI. These contain a discussion of the photo-elastic effect, *i.e.* the phenomenon of artificial birefringence due to stress or strain in an otherwise optically homogeneous medium, and also how this phenomenon is utilised to obtain information about complex stress systems which are not amenable to mathematical analysis.

Another chapter is devoted to the extension of the photoelastic technique to three-dimensional work, whilst the remainder of the book will perhaps prove most useful to any reader intending to set up a photoelastic apparatus. These final chapters contain discussions of the design of a suitable optical set-up, of the properties of model materials and, perhaps most important to readers unfamiliar with this field, of a number of worked-out examples illustrated by excellent photographs of the relevant stress patterns.

This book is primarily addressed to either actual or potential practitioners of the art and science of photoelasticity, but it should also be of interest to applied mathematicians with interests in the theory of elasticity. This book will show them how engineers and physicists deal with problems which the mathematical theory cannot deal with. E. K. FRANKL.

Mechanics via the Calculus. By P. W. NORRIS and W. S. LEGGE. 3rd edition. Pp. xii, 367. 16s. 1950. (Cleaver-Hume Press)

This book was first published in 1923 by Messrs. Longmans Green; a second edition followed in 1930, which was later reprinted, so for many years the volume has found a place in the teaching of applied mathematics in schools and colleges. A third edition now appears under interesting circumstances; the Cleaver-Hume Press have reproduced the book by the use of photolithography and made it available at a price that, they claim, would otherwise have been impossible. A few new examples have been added, and there is a new chapter on vectors and two appendices on products of inertia and instantaneous centres.

The book is primarily designed for the mathematical specialists in Grammar Schools, and for those in Colleges and Universities attending courses of pass degree standard. It covers adequately the usual topics in statics, dynamics of a particle and a rigid body, and hydrostatics. The many worked examples throughout the book are invaluable, and there are ample examples, some 550, set for solution.

The new chapter on vectors, which unfortunately has several misprints, deals with the elements of vector algebra, scalar and vector products and triple products, and the derivative of a vector with respect to a scalar, including the usual applications to velocity and acceleration. The moment of a force "about a point" and about an axis are expressed vectorially, the former as a vector and the latter as a scalar, which will be confusing to some students. The authors have intended this chapter as an introduction to the subject, but it is doubtful whether they have been wise to include it, especially as throughout the book vector methods have not been used. One can only hope that readers will attempt to use vectors in the solution of some of the problems appearing earlier in the book.

The new appendix on products of inertia and principal axes is a very necessary addition and could well have been fuller. It is unfortunate, however,

that the first example dealt with, the product of inertia of a rectangular lamina, is not evaluated starting from the definition; to assume an equimomental system, when previous work on such systems deals only with moments of inertia, is unsound.

It may be that the method of reproduction of this edition has prevented any thoroughgoing revision of the text, but one would have welcomed the removal of a few blemishes. For example, in the theory of attractions the potential is hardly mentioned and the relation between gravitational intensity and potential not sufficiently stressed. Again the opening paragraph of the chapter on hydrostatics might be clearer; what is meant by "In the case of a rigid body, however, the stresses acting may have components normal to any plane drawn to the solid"? Also the term "centre of inertia" is used throughout rather than "centre of mass", and "centre of gravity" and "centre of inertia" are used interchangeably; for example, at the top of p. 178 both terms are used in one sentence. There is no mention of weight in dealing with centres of gravity.

But these are small matters. Perhaps the greatest improvement that could have been effected would be to have included a serious discussion of the basis of mechanics, which is very important for readers of a book of this standard. As it is, one fears that students working through this book will have been helped admirably to solve certain problems in applied mathematics, and have fashioned perhaps an impressive technique, but will not understand the foundations of the subject as a science, nor appreciate how it has grown in history. For Newton's laws of motion are dismissed very briefly, and later one short paragraph suffices for D'Alembert's Principle. A final chapter discussing these basic principles would have considerably enhanced the value of the book.

J. TOPPING.

Mechanics: Statics and Dynamics. By M. SCOTT. Pp. xi, 394. 38s. 6d. 1949. (McGraw-Hill)

The author is a professor of physics in the Pennsylvania State College, and has written his book to satisfy the needs of his own students and department. In this country such a book would be described as covering the ground of a first year university course in mechanics.

The book is divided distinctly into two parts, the first section dealing with statics, including a little of the elements of elasticity, whilst the second section is devoted entirely to dynamics. Both sections start from first principles, although it is assumed that the reader has already some acquaintance with the subject. The vector notation is used wherever possible in the volume, and for this purpose the statics section commences with an introductory chapter on very elementary vector algebra. This section then proceeds to deal with the usual topics, including the properties of forces, centres of gravity, and friction, etc. Also, there are chapters on the elements of elasticity, gravitational potential, and stability of equilibrium. The second section, apart from the last chapter, deals entirely with two-dimensional dynamics of particles and rigid bodies.

In my opinion the author has attempted to cover too much ground in this volume. The result of this is evident, even in the first few chapters on the equilibrium of forces and centres of gravity; these accounts are far too sketchy. The chapter on friction also suffers in the same way, and certainly needs support in the nature of many more worked examples. The dynamics, on the other hand, is given a fuller presentation, and is the more interesting in consequence. Each chapter is accompanied by a set of examples, mostly with a bias towards the arithmetic type of question.

The appearance of the book is very attractive, and the printing, apart from the two minor slips which I describe below, is excellent throughout.

(i) Equation (20.31) on page 295 should read :

$$\delta = \frac{K}{|V|} = 1 + \frac{C}{|V|}, \text{ instead of } \delta = \frac{K}{|V|} = 1 + C.$$

(ii) In equations (*a*) and (*b*) of (21.75) on page 323 replace $\geqslant \mu$ and $< \mu$ by $> \mu$ and $\leqslant \mu$ respectively.

In conclusion I should state that there is a list of all the main symbols used in the course of the work together with their meanings printed towards the front of the book, and an adequate index of the subject-matter tabulated, as usual, at the back of the book. J. W.

Applied Mechanics : Statics. By G. W. HOUSNER and D. E. HUDSON. Pp. v, 220. 1949. (D. Van Nostrand Co. Inc.)

This book on practical statics is written by two engineers of the California Institute of Technology for the use of engineering students. The principal aim of the book is " to form a logical transition from the elements of mechanics as studied in the general physics course to the more advanced courses in engineering ". Consequently, it does not take a great deal of imagination to realise that here we have a book with examples taken from everyday engineering problems.

In all there are seven chapters. The first three of these are concerned with " Composition and Resolution of Force Systems ", " Equilibrium of Force Systems ", " Principle of Virtual Displacements ", and, according to the authors, are intended primarily to give " a concise treatment of the general principles of statics ". Consequently, true to this statement, we have these elemental principles expounded always with an eye to the applications and accompanied with neat isometric diagrams. The student is guided from the start to seek out statical principles from schematic views, such as in the case of the one cylinder of a reciprocating internal combustion engine.

The remaining four chapters " contain a large number of applications of statics to various engineering fields such as Structures, Fluid Mechanics and Machine Design ". For example, in Chapter IV, after centres of gravity, moments and products of inertia have been discussed, we find harder practical problems of evaluation. The chapter concludes with force distributions leading directly to hydrostatic problems. In Chapter V we find an interesting discussion on friction in all its practical forms, while Chapter VI is concerned with the " Elementary Theory of Statically Determinate Structures ". The final chapter gives a short introduction to " Statically Indeterminate Structures ". To conclude, there are two appendices, concerned respectively with a bibliography and tabulated statical properties of prisms of various sections and solid bodies. There is a list of answers to the 211 problems set at various stages in the body of the work, and a very satisfactory index.

The book, I feel, will certainly prove useful to those students who are reading for an engineering degree, and also for those pure science people who have an eye to future work in industry.

In conclusion, I must state that from a typographical point of view the book is delightful to read, with the printing and diagrams agreeably clear.

J. W.

Progressive Mathematics. By P. CLYNE. Pp. xii, 270. 15s. 1950. (Chapman & Hall)

The chief novel feature of this book is a conversational style, including interpolated dialogues between author and student, often in a humorous vein.

These undoubtedly make the book easy to read, and will appeal to the class of student whom the author has chiefly in mind, namely, those whose aim is a Higher National Certificate. The necessity of covering a wide range of Mathematics in the short time available to part-time students raises special problems of presentation and selection. This book takes the Calculus as far as the simpler Differential Equations, with a chapter on Conic Sections and brief treatments of Complex Numbers, Partial Differentiation and Fourier Series. Some topics, including permutations and combinations, curves such as the cycloid and epicycloid, polar coordinates and determinants, are dealt with in appendices.

It would be pedantic to criticise the presentation from the standpoint of rigour. Such a course must inevitably use short-cuts to get results quickly. The difficulties of limits and orders of infinitesimals are here largely by-passed. There is a preliminary chapter on Indeterminate Quantities written in popular terms ; we read, for instance, " we must be careful not to divide both . . . by 0 unless it is the same sized 0 ". In the necessary differentiations from first principles the limits are properly obtained, h and k replacing the usual δx and δy, which nowhere appear. Similarly when integration is reached the limit $dA/dx = y$ is reasonably demonstrated ; from then on an element is given a width dx. Integration by substitution is treated as obvious and introduced *en passant* ; it seems doubtful whether the same easy introduction of ds^2 in Cartesians and polars is justifiable.

Accepting all this, one must still take exception to some particular " proofs " and statements. The limit $\sin\theta/\theta$ is obtained in the usual way ; in the following paragraph it is used to obtain the area of a circle, via a regular polygon : but the formula πr^2 has already been used in proving the limit. In spite of warnings to the student against false converses in relation to maxima and minima, we are told " d^2y/dx^2 is negative when y is a maximum . . . positive when y is a minimum ". Two pages later this is disproved by the example of $y = x^4$, but we still read " even though $d^2y/dx^2 = 0$, y is a point of inflexion only when $d^3y/dx^3 \neq 0$ ". Newton's formula for ρ is based on a diagram which assumes symmetry of the curve ; it might well have been left to the next chapter and made an application of Maclaurin's series. The treatment of convergence is not satisfactory ; some mention of absolute convergence is necessary if results proved for positive terms are to be applied to the standard power series. It is falsely stated in § 77 that if a power series for y is convergent, so is the series obtained by differentiating term by term. Since the author regards a finitely oscillating series as in some sense convergent, he presumably would not admit the instance of the series for $\log(1+x)$ and for $1/(1+x)$, at $x = 1$, as disproving his theorem, but how would he dispose of the two series

$$x + \frac{x^2}{1 \cdot 2} + \frac{x^3}{2 \cdot 3} + \ldots \quad \text{and} \quad 1 + \frac{x}{1} + \frac{x^2}{2} + \ldots$$

at $x = 1$? The fallacy in his " proof " is the usual one of failure to distinguish between a ratio < 1 and a ratio $< c < 1$. Principal values of the inverse trigonometrical functions are never defined, and no justification is given for the choice of sign in the derivative of $\sin^{-1} x$. The brief chapter on Partial Differentiation includes the statement : " Total Differential.—This is defined as dz/dx where $z = f(x, y)$ and y is a function of x ".

It is hoped that these criticisms will not appear to be other than constructive. Since the *style* of the book is intended to appeal to the student who is not qualified to judge the excellence of its *matter*, a review would seem to be the proper place in which to draw attention to some defects in the latter. In the reviewer's opinion, the book needs revision and enlargement ; but perhaps

it is impossible to treat Mathematics adequately and retain popularity of style.

There are a number of minor misprints. Some more important ones are: Fig. 38 is out of place and belongs to Ex. 5, two pages later; Ex. VIII, 34, requires r for v and should read "the ratio r/R is $1/\sqrt{e}$; p. 196, line 10, dy/dz should be dy/dx; p. 205, worked example, a minus sign is omitted from R.H.S. of the D.E. and in the first line of p. 206; p. 247, line 12 has ds for dr on L.H.S. and a minus sign omitted from R.H.S. The answer to Ex. I, 8, should be $1/\sqrt{6}$ and not zero. Finally, it would surely be desirable to modify Ex. XV, 30, in which the charge on a condenser is allowed to fall to one electron; the physical law involved is of a statistical nature. C. G. P.

Designing and Making. By W. W. SAWYER and L. G. SRAWLEY. Pp. xiii, 192. 9s. 6d. 1950. (Blackwell)

The rise of the Secondary Modern School has brought with it the cry for mathematics which has meaning to the pupil, or is capable of immediate application to practical situations. The fashion of the moment is to select material which is likely to be of interest for its own sake and to extract some mathematics from it. This is the reverse of the treatment we have become accustomed to, in which the principle comes first, the application following, often almost as an afterthought; here the application is paramount.

As we have come to expect from Mr. Sawyer, the choice of material is excellent. Aeroplanes, railways, bicycles, gadgets, gardens, sports and pastimes; if we have seen some of it before, yet there is enough that is new to be worth while. The boy or girl who is fortunate enough to obtain a copy of this book will spend pleasant hours reading and making, if not designing.

Those who have used this method in teaching will know its difficulties. The extraction ratio is low; what does come out is patchy. One would need a great pile of books of this type from which to develop a systematic course of elementary mathematics, even at this level. Nevertheless, as a stimulus to interest in mathematics, as an answer to the eternal *why*, or as a source of occasional material for the teacher, the book is invaluable. I. R. V.

Harvard University. The Annals of the Computation Laboratory. Volumes 18, 19: Tables of Generalized Sine- and Cosine-Integral Functions. Part I: Pp. xxxviii + 462; Part II: Pp. viii + 560. $8'' \times 10\frac{1}{2}''$. 1949. 55s. net each volume. (Harvard University Press; London, Geoffrey Cumberlege)

In these two volumes the following six functions are tabulated together on each page:

$$S(a,x)=\int_0^x \frac{\sin u}{u}\,dt, \qquad C(a,x)=\int_0^x \frac{1-\cos u}{u}\,dt,$$

$$Ss(a,x)=\int_0^x \frac{\sin u \sin t}{u}\,dt, \qquad Cs(a,x)=\int_0^x \frac{\cos u \sin t}{u}\,dt,$$

$$Sc(a,x)=\int_0^x \frac{\sin u \cos t}{u}\,dt, \qquad Cc(a,x)=\int_0^x \frac{\cos u\,(1-\cos t)}{u}\,dt.$$

In these formulae, $u=\sqrt{(a^2+t^2)}$. The functions $S(a,x)$ and $C(a,x)$ are generalizations of the ordinary sine- and cosine-integrals

$$Si(x)=\int_0^x \frac{\sin t}{t}\,dt, \qquad Ci(x)=\int_\infty^x \frac{\cos t}{t}\,dt,$$

since, when $a=0$,

$$S(0,x)=Si(x), \qquad C(0,x)=\gamma+\ln x - Ci(x).$$

If the integral for $S(a, x)$ is written in the form

$$\int_1^{\sqrt{(1+x^2/a^2)}} \frac{\sin at}{\sqrt{(t^2-1)}}\,dt$$

and is compared with the known formula for the Bessel function of zero order

$$J_0(z)=\frac{2}{\pi}\int_1^{\infty} \frac{\sin zt}{\sqrt{(t^2-1)}}\,dt,$$

it is seen that $S(a, x)$ may be regarded as an " incomplete " Bessel function, with $S(a, \infty)=\frac{1}{2}\pi J_0(a)$. Similarly, $C(a, x)$ is related to $Y_0(a)$. The four composite functions appear to be simpler ; for example :

$$2Sc(a, x)=Si(\sqrt{a^2+x^2}+x)-Si(\sqrt{a^2+x^2}-x).$$

The tables were made to meet the demands of the theory of antennae as developed during wartime research ; the preliminaries to Part I include a note on *Applications* by R. W. P. King. The Automatic Sequence Controlled Calculator, working under Navy contract, performed the numerical work and produced typescript which was used for photographic reproduction. Each function is given to *six* decimals throughout and the tabulations are collected under values of a arranged in ascending order of magnitude. The ranges of argument are as follows :

Part I		*Part II*	
x	a	x	a
0(·01)0·99	0(·01)0·99	0(·05)4·95	2(·05)4·95
0(·02)0·98	1(·02)1·98 1·05(·1)1·95	0(·1)9·9	5(·1)9·9
1(·02)1·98	0(·02)1·98 0·05(·1)1·95	0(·2)25	10(·2)25
2(·05)4·95	0(·05)1·95	5(·1)9·9	2(·1)4·9
5(·1)9·9	0(·1)1·9	10(·2)25	2(·2)9·8
10(·2)25	0(·2)1·8		

It will be observed that the tables are far from being homogeneous.

Some reflections are prompted by examination of this material.

1. *Double-entry tables : choice of arguments and compactness.* In this project " the staff of the Computation Laboratory has undertaken to present as complete and useful a table . . . as can be included within the scope of a thousand pages ". They have certainly filled 1000 pages and more. A table of double entry is inevitably rather bulky, and interpolation in it is troublesome. That being so, it is particularly desirable that the best choice of arguments be made and justified, and that adequate aids to interpolation be provided ; the reviewer is not convinced that either matter has in this case been given full attention. Suppose, for example, that the generalized sine-integral is tabulated not against a and x, but against a and $h=x/a$, it can be shown that

$$S(a, h)=\int_0^{\sinh^{-1}h} \sin(a\cosh t)\,dt=\sum_{r=0}^{\infty}\frac{(-1)^r a^{2r+1}}{(2r+1)!}\int_0^h (1+t^2)^r\,dt$$

(a rather simple computing formula for small a and h). For each $a>1$, the range of h is less than that of x, and it is possible that more homogeneity might be obtained. The text does not make it clear why a and x are the most suitable parameters.

2. *Methods of computation.* It is perhaps a pity that, as computing equipment becomes more powerful, the mathematical methods in use sometimes

become more pedestrian. The machine at Harvard seems to thrive on a power series in its most primitive form, and where this leads to the most efficient computation its employment may be justified; but in a standard volume of tables, one would like to see exhibited more of the mathematical structure of the functions in question. For example, a connection is expected between $S(a, x)$ and the Bessel functions of half-odd-integer order $J_{n+\frac{1}{2}}(z)$, yet on page xxviii of the introduction to Part I, the explicit expressions for the latter functions are, in effect, written out in full without their identity being apparently recognised. If a series is wanted, it is possible to prove that

$$S(a, x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!}\, a^{2r} \int_0^x \sqrt{\left(\frac{\pi}{2t}\right)}\, J_{r+\frac{1}{2}}(t) \cdot \frac{dt}{(2t)^r},$$

and this, in conjunction with existing tables of the spherical Bessel functions, might yield an easy calculation for small values of a.

3. *Methods of presentation.* These volumes, like many other American books of tables, are made by printing from a photograph of the typed page, and there is much to be said in favour of this process, especially when conventional printing establishments are overworked. One of its advantages, when hundreds of pages are alike, is that considerable use may be made of rules and other embellishments, without the additional cost and complication of setting these up in pieces on every page, as in ordinary printing from type. It gives pleasure to the reviewer to see a second argument column on the right of each page (a practice which might well be imitated by those who design railway timetables for display in stations). On the other hand, he would like to see omitted the digits before the decimal points (except the first in a group), and to see included an occasional + to mark more certainly the places at which there are sign changes.

Criticism of these volumes is helpfully intended, and must not obscure the feeling of debt owed to American computing establishments, which are producing many basic tables, themselves already of value in making other tables.

C. W. JONES.

1654. Since that night on the roof when Flora had said to him, "There is a God", he had felt a certain restlessness. It was as if he were presented with a vast and complex algebraic problem, and Flora had said the answer was $x = 349$. But to know the answer was one thing; to work it out through four pages of mystifying equations was another. The fact that the answer was provided in a text-book prepared by the first mathematicians of the times would not make any difference. It would not satisfy Tom's enquiring mind. —Lin Yutang, *Chinatown Family*, p. 100 (Heinemann, 1949). [Per Mr. A. R. Pargeter.]

1655. But besides the value of Mathematical Studies in Education, as a perfect example and complete exercise of demonstrative reasoning; Mathematical Truths have this additional recommendation, that they have always been referred to, by each successive generation of thoughtful and cultivated men, as examples of truth and of demonstration; and have thus become standard points of reference, among cultivated men, whenever they speak of truth, knowledge, or proof. . . . The recollection of the truths of Elementary Geometry has, in all ages, given a meaning and a reality to the best attempts to explain man's power of arriving at truth.—*Of a Liberal Education*, etc., by W. Whewell, D.D., Master of Trinity College, Cambridge. [Per Mr. B. A. Swinden.]

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